Periodic Pólya urns and an application to Young tableaux

JCB 02/2019

Michael Wallner

joint work with Cyril Banderier and Philippe Marchal

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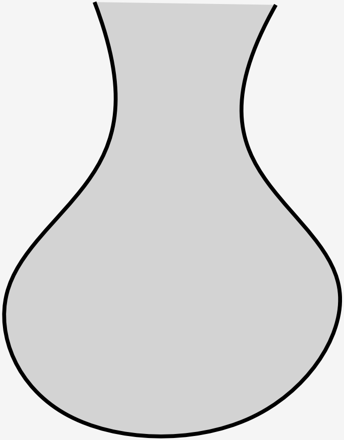
February 13^{th}, 2019

Based on the paper:
Periodic Pólya urns and an application to Young tableaux, AofA 2018

Introduction

What are Pólya urns?



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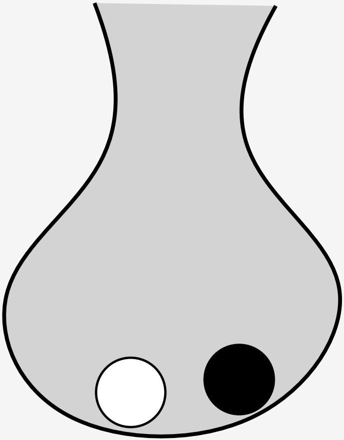

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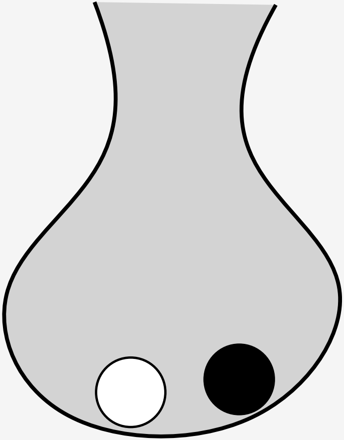

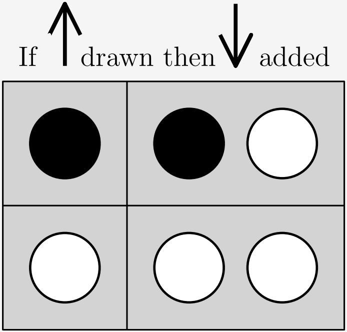

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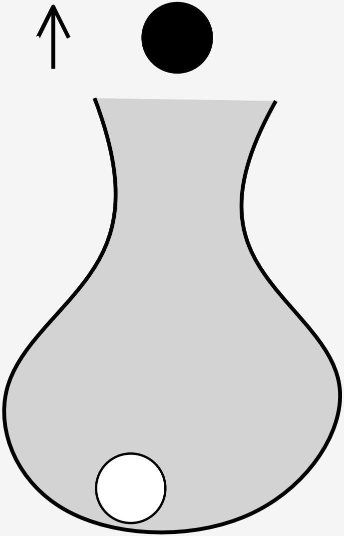

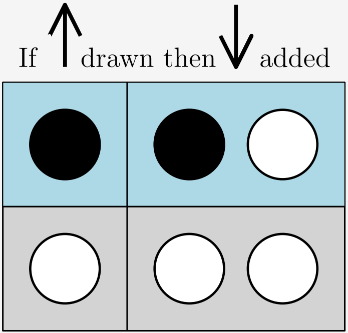

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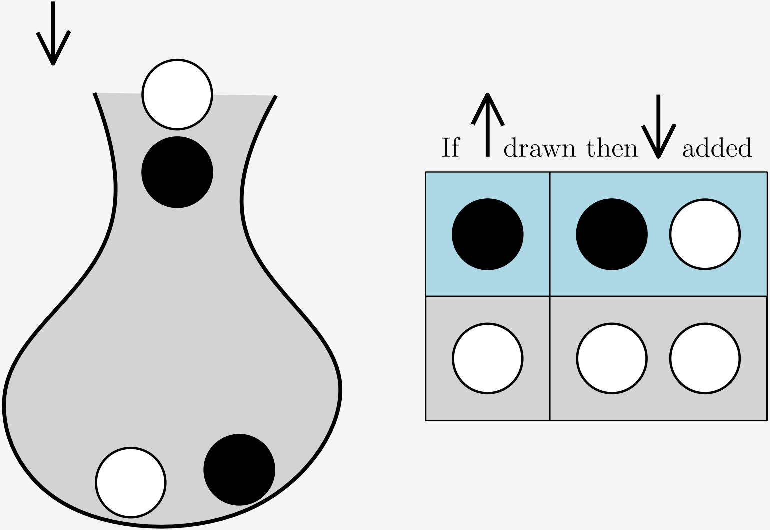

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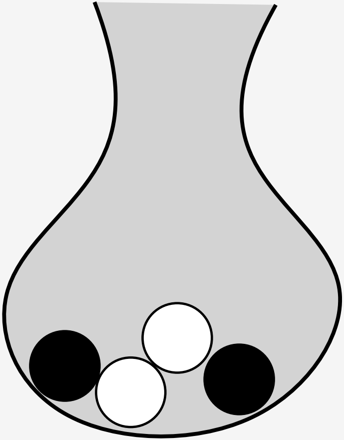

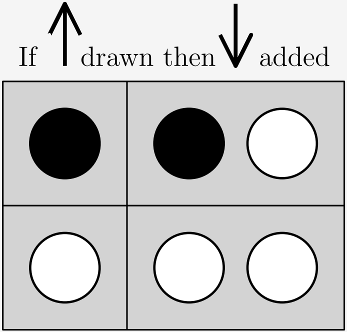

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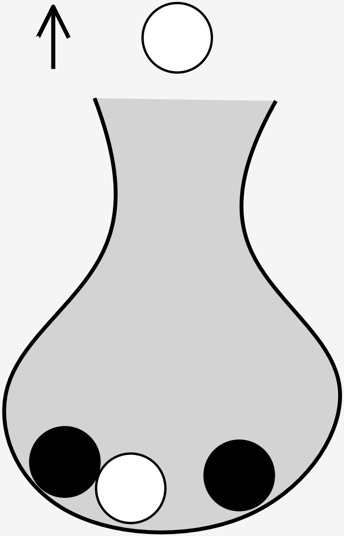

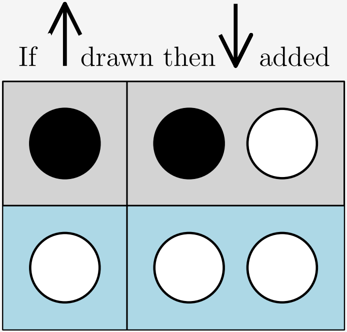

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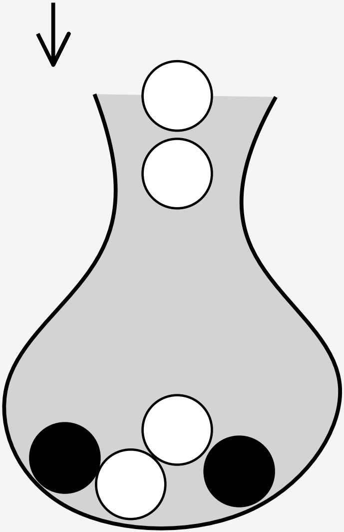

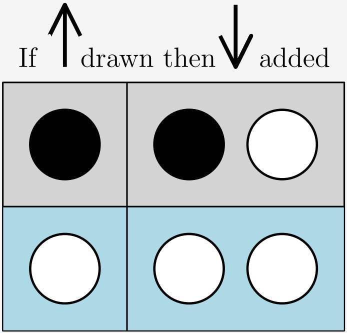

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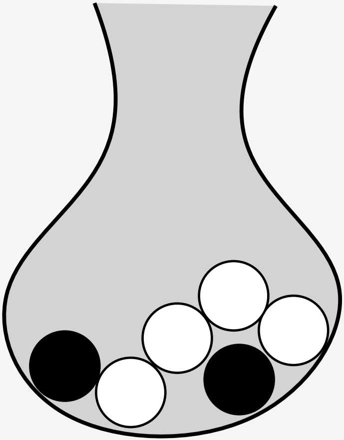

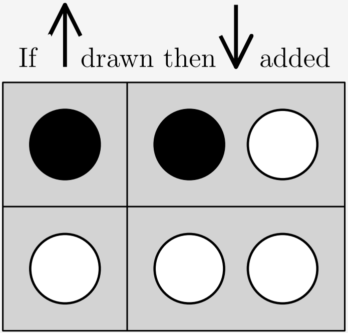

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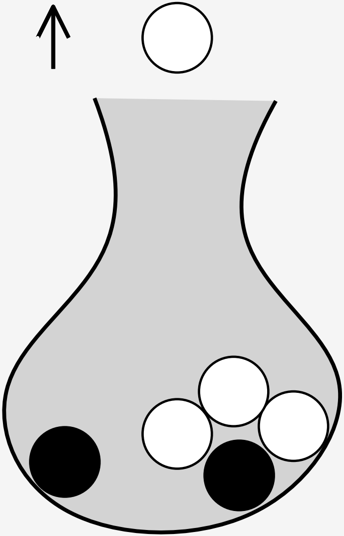

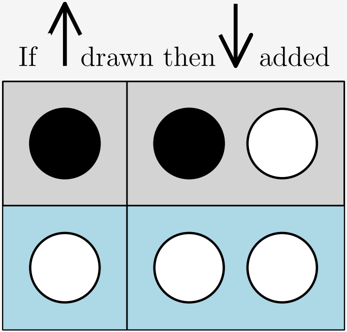

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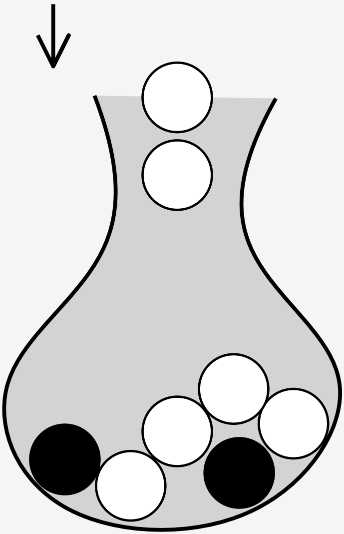

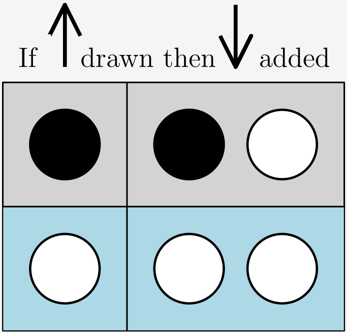

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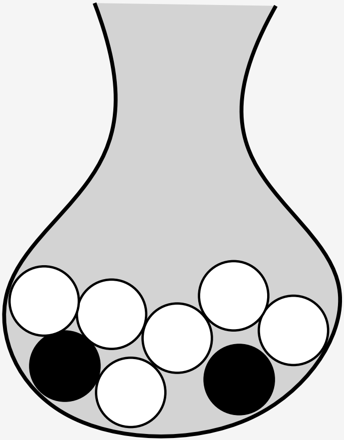

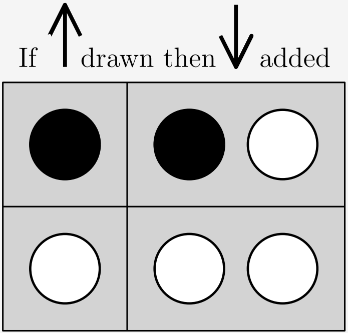

A formal definition

Replacement matrix

Let $a, b, c, d \in \mathbb{Z}_{\geq 0}$.

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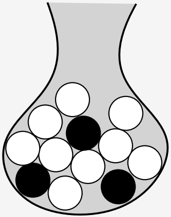

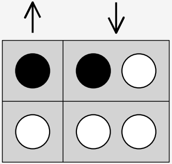

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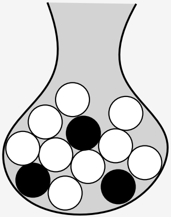

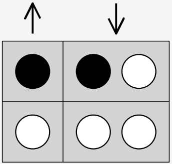

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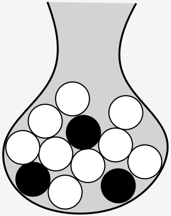

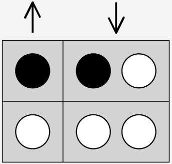

- Balanced urn: $K := a + b = c + d$ (above $K = 2$)
- Initial b_0 black ($\bullet$) and w_0 white ($\circ$)
- After n steps $b_0 + w_0 + Kn$ balls in the urn (deterministic!)

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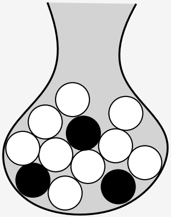

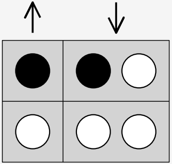

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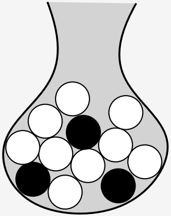

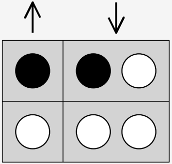

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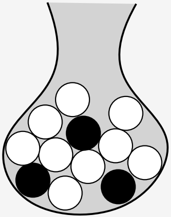

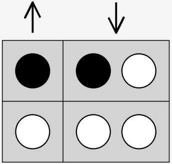

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Famous urn models [Flajolet, Dumas, Puyhaubert 2006]

Contagion urn (Pólya)	Adverse-campaign urn (Friedman)	Coupon collector's urn
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

Vast number of applications

- [Pólya, Eggenberger 1923-1930]: Disease infections $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$
- [Rivest 2012]: How to check election results? How to be sure your vote was counted?
- [Fanti, Viswanath 2017]: Deanonymization in Bitcoin's peer-to-peer network
- [Smythe, Mahmoud 1994, Holmgren, Janson 2016]: m-ary search trees
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Vivid field:

- many tools (martingales, analytic combinatorics, contraction, stochastic approximation, ...),
- many experts (see above, and more),
- many challenges: more colors, subset samplings, non tenable (non trivial positivity constraint), non time homogeneous, ...

Periodic Pólya urns

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The *Young–Pólya urn* is a Pólya urn of period 2 with

replacement matrix $M_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for every odd step,

and replacement matrix $M_2 := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ for every even step.

Caveat: different from multidrawing models

Histories

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Histories of length n: A sequence of n drawings/evolutions.
$h_{n,k,\ell}$: Number of histories of length n, from (b_0, w_0) to (k, ℓ).

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Generating function of histories:

$$H(x,y,z) = \sum_{n\geq 0} H_n(x,y)\frac{z^n}{n!} = \sum_{n,k,\ell\geq 0} h_{n,k,\ell}x^k y^\ell \frac{z^n}{n!}.$$

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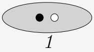

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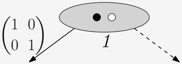

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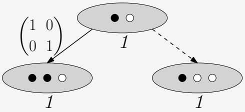

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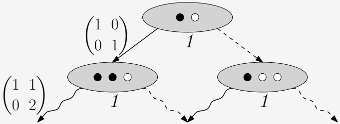

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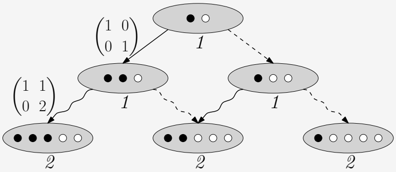

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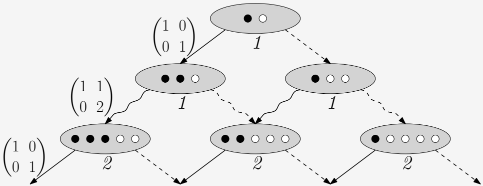

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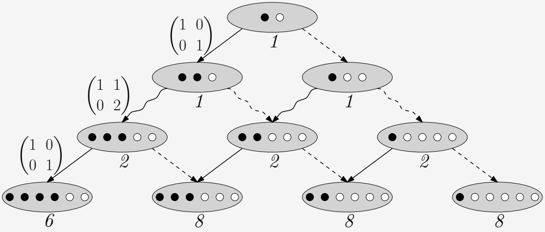

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Distributions

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Central question?

How does the composition of the urn look like after n steps?

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Panacea! Histories generating function

$$H(x,y,z) = xy + \left(xy^2 + x^2y\right) z + \left(2xy^4 + 2x^2y^3 + 2x^3y^2\right) \frac{z^2}{2} + \dots$$

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Panacea! Histories generating function

$$H(x,y,z) = xy + \left(xy^2 + x^2y\right) z + \left(2xy^4 + 2x^2y^3 + 2x^3y^2\right) \frac{z^2}{2} + \dots$$

Balanced urn

- Homogeneous polynomials $H_n(x,y)$
- Deterministic number of balls

$\Rightarrow$ Distribution B_n of black balls after n steps

Distribution of the urn

Central question?

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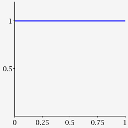
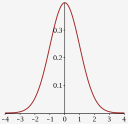
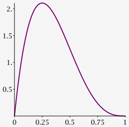
Many known distributions for (non-periodic) urn models



Uniform distribution



Normal distribution



Beta distribution $\mathcal{B}(2,4)$

Periodic Pólya urns extend the family

Generalized Gamma distribution GenGamma(α, β)

Let $\alpha, \beta > 0$ be real, then the density function with support $(0, +\infty)$ is

$$f(x; \alpha, \beta) := \frac{\beta\, x^{\alpha-1} \exp(-x^{\beta})}{\Gamma\left(\alpha/\beta\right)},$$

where Γ is the classical Gamma function $\Gamma(z) := \int_0^\infty t^{z-1} \exp(-t)\, dt$.

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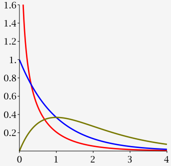
Some examples with red $\alpha = 0.5$, blue $\alpha = 1$, and green $\alpha = 2$:



$\beta = 1$

Gamma distributions

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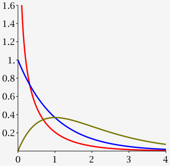

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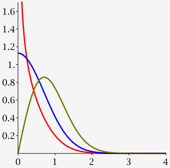

$\beta = 2$

Half-normal for $\alpha = 1$

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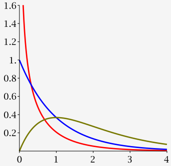

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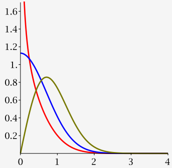
Gamma distributions



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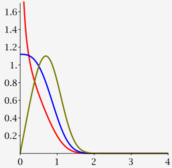

$\beta = 3$

This talk!

Periodic Pólya urns extend the family

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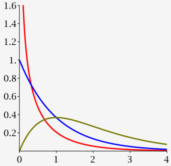

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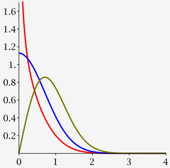

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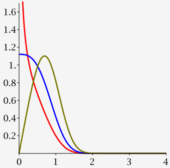


$\beta = 3$

This talk!

- [Janson 2010]: area of the supremum process of the Brownian motion
- [Peköz, Röllin, Ross 2016]: preferential attachments in graphs
- [Khodabin, Ahmadabadi 2010]: generalization of special functions

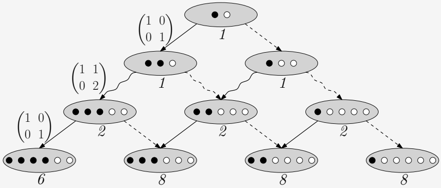
The number of black balls

Black balls in Young–Pólya urns

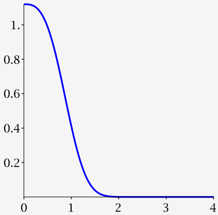
Theorem

The normalized random variable $\frac{2^{2/3}}{3}\frac{B_n}{n^{2/3}}$ *of the number of black balls in a Young–Pólya urn converges in law to a generalized Gamma distribution:*

$$\frac{2^{2/3}}{3}\frac{B_n}{n^{2/3}} \quad \xrightarrow{\mathcal{L}} \quad \mathrm{GenGamma}\,(1,3)\,.$$



The evolution of the Young–Pólya urn



Density: $f(x;1,3) = \frac{3\,e^{-x^3}}{\Gamma(1/3)}$

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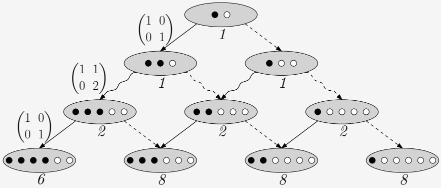

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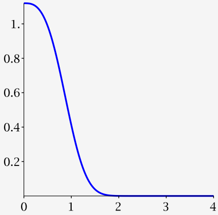

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1. Classical Pólya urn $\begin{pmatrix}1&0\\0&1\end{pmatrix}$: $c_1\frac{B_{1,n}}{n} \quad \xrightarrow{\mathcal{L}} \quad$ uniform distribution
2. Classical Pólya urn $\begin{pmatrix}1&1\\0&2\end{pmatrix}$: $c_2\frac{B_{2,n}}{\sqrt{n}} \quad \xrightarrow{\mathcal{L}} \quad$ half-normal distribution

Capture the evolution of the urn I

1. Split into even and odd steps

$$H_e(x,y,z) := \sum_{n\geq 0} H_{2n}(x,y)\frac{z^{2n}}{(2n)!}$$

$$H_o(x,y,z) := \sum_{n\geq 0} H_{2n+1}(x,y)\frac{z^{2n+1}}{(2n+1)!}$$

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2. Model the evolution using differential operators

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\mathcal{D}_1 := x^2\partial_x + y^2\partial_y \qquad \mathcal{D}_2 := x^2y\partial_x + y^3\partial_y$$

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3. Link odd and even steps

$$\partial_z H_o(x,y,z) = \mathcal{D}_1 H_e(x,y,z) \qquad \partial_z H_e(x,y,z) = \mathcal{D}_2 H_o(x,y,z)$$

Capture the evolution of the urn II

4. Eliminate the (redundant) y variable (white balls) due to

$$\text{number of balls after } n \text{ steps} = 2 + n + \left\lfloor \frac{n}{2} \right\rfloor .$$

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This gives

$$\begin{cases} x\partial_x H_e(x,y,z) + y\partial_y H_e(x,y,z) = 2H_e(x,y,z) + \frac{3}{2} z\partial_z H_e(x,y,z), \\ x\partial_x H_o(x,y,z) + y\partial_y H_o(x,y,z) = \frac{3}{2} H_o(x,y,z) + \frac{3}{2} z\partial_z H_o(x,y,z). \end{cases}$$

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5. Algebra! Eliminate $\partial_y H_e$ and $\partial_y H_o$, then substitute $y = 1$
(We define $H(x,z) := H(x,1,z)$, $H_e(x,z) := H_e(x,1,z)$, and $H_o(x,z) := H_o(x,1,z)$)

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Moreover, they satisfy linear *differential equations, i.e., they are* D-finite.

Urns are D-finite!

Let $\widetilde{H}(x,z) := \sum_{n\geq 0} \frac{H_n(x)}{H_n(1)} z^n$ be the probability generating function. Then we have

$$L\widetilde{H}(x,z) = (\cdots \partial_z^3 + \cdots \partial_z^2 + \cdots \partial_z)\widetilde{H}(x,z) = 0.$$

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$$\begin{aligned}
L = {} & 9\,z(z-1)(z+1)\left(x^3z^2+2x^3z+3x^2-3x+1\right)\left(x^3z^2-2x^3z+3x^2-3x+1\right)\left(15x^7z^6+36x^7z^5+45x^7z^4-3x^6z^5+15x^6z^4+180x^6z^3-24x^5z^4-210x^5z^3\right.\\
& +8x^4z^4-108x^6z+63x^5z^2+90x^4z^3+180x^5z-108x^4z^2-10x^3z^3-81x^5-135x^4z+81x^3z^2+189x^4+54x^3z-30x^2z^2-189x^3-9x^2z+5xz^2\\
& \left.+99x^2-2zx-27x+z+3\right)\partial_z^3+3\left(375x^{13}z^{12}+1188x^{13}z^{11}+912x^{13}z^{10}-84x^{12}z^{11}-2646x^{13}z^9+1419x^{12}z^{10}-4995x^{13}z^8+9822x^{12}z^9-1734x^{11}z^{10}\right.\\
& -486x^{13}z^7+5853x^{12}z^8-10974x^{11}z^9+578x^{10}z^{10}+1620x^{13}z^6-24138x^{12}z^7-1902x^{11}z^8+4506x^{10}z^9-7938x^{12}z^6+53940x^{11}z^7-5102x^{10}z^8\\
& -424x^9z^9+24138x^{12}z^5+5292x^{11}z^6-65778x^{10}z^7+5958x^9z^8+3240x^{12}z^4-62748x^{11}z^5+11358x^{10}z^6+49444x^9z^7-2472x^8z^8-7776x^{12}z^3\\
& +13932x^{11}z^4+102168x^{10}z^5-29508x^9z^6-21948x^8z^7+412x^7z^8+27054x^{11}z^3-46224x^{10}z^4-120504x^9z^5+33384x^8z^6+4918x^7z^7-8748x^{11}z^2\\
& -55242x^{10}z^3+69579x^9z^4+102294x^8z^5-22658x^7z^6-360x^6z^7+25434x^{10}z^2+85158x^9z^3-77346x^8z^4-61030x^7z^5+9456x^6z^6+2430x^{10}z\\
& -46332x^9z^2-104976x^8z^3+71694x^7z^4+24626x^6z^5-2286x^5z^6-13770x^9z+76869x^8z^2+100998x^7z^3-52878x^6z^4-6102x^5z^5+254x^4z^6+5103x^9\\
& +35964x^8z-109242x^7z^2-72612x^6z^3+29151x^5z^4+718x^4z^5-22113x^8-56646x^7z+116856x^6z^2+37458x^5z^3-11534x^4z^4-8x^3z^5+44226x^7\\
& +59346x^6z-89910x^5z^2-13098x^4z^3+3144x^3z^4-53298x^6-43092x^5z+49137x^4z^2+2698x^3z^3-540x^2z^4+42525x^5+21906x^4z-18726x^3z^2\\
& \left.-144x^2z^3+45xz^4-23247x^4-7674x^3z+4758x^2z^2-66xz^3+8694x^3+1764x^2z-728xz^2+12z^3-2142x^2-238zx+51z^2+315x+14z-21\right)\partial_z^2\\
& +2\left(1020x^{13}z^{11}+4032x^{13}z^{10}+7461x^{13}z^9-276x^{12}z^{10}+972x^{13}z^8+1317x^{12}z^9-9315x^{13}z^7+29340x^{12}z^8-2559x^{11}z^9-5346x^{13}z^6+27990x^{12}z^7\right.\\
& -34260x^{11}z^8+853x^{10}z^9-52974x^{12}z^6-19935x^{11}z^7+14352x^{10}z^8-40743x^{12}z^5+113634x^{11}z^6-5916x^{10}z^7-1466x^9z^8+40338x^{12}z^4+25839x^{11}z^5\\
& -127818x^{10}z^6+13083x^9z^7+19440x^{12}z^3-150660x^{11}z^4+40608x^{10}z^5+89886x^9z^6-5442x^8z^7-11664x^{12}z^2-4617x^{11}z^3+240894x^{10}z^4-80883x^9z^5\\
& -38142x^8z^6+907x^7z^7+58806x^{11}z^2-92340x^{10}z^3-206514x^9z^4+69570x^8z^5+8198x^7z^6-17496x^{11}z-115182x^{10}z^2+227124x^9z^3+93150x^8z^4\\
& -38517x^7z^5-526x^6z^6+58563x^{10}z+127008x^9z^2-289710x^8z^3-12354x^7z^4+14154x^6z^5-10206x^{10}-105462x^9z-93312x^8z^2+232929x^7z^3-9330x^6z^4\\
& -3231x^5z^5+27216x^9+139482x^8z+50922x^7z^2-122508x^6z^3+5958x^5z^4+359x^4z^5-20412x^8-150903x^7z-20358x^6z^2+41499x^5z^3-1632x^4z^4\\
& -19278x^7+134298x^6z+4050x^5z^2-8526x^4z^3+194x^3z^4+55566x^6-94770x^5z+1350x^4z^2+948x^3z^3-57834x^5+50715x^4z-1416x^3z^2-60x^2z^3\\
& \left.+35910x^4-19620x^3z+540x^2z^2+5xz^3-14490x^3+5148x^2z-110xz^2+3780x^2-819zx+10z^2-588x+60z+42\right)\partial_z+17010x^{10}-61236x^9+102060x^8\\
& -103194x^7+70308x^6-34020x^5+11970x^4-3024x^3+504x^2-42x+12x\left(270x^6-261x^5+126x^4+21x^3-69x^2+30x-5\right)\left(3x^2-3x+1\right)^2z\\
& +2x\left(3x^2-3x+1\right)\left(3240x^9-2673x^8-6129x^7+16254x^6-16101x^5+8010x^4-1923x^3+78x^2+45x-5\right)z^2\\
& +4x^3\left(3x^2-3x+1\right)\left(1134x^7-4995x^6+5886x^5-2841x^4+129x^3+366x^2-159x+28\right)z^3\\
& -6x^4\left(3x^2-3x+1\right)\left(1485x^6-468x^5-927x^4+1150x^3-623x^2+162x-27\right)z^4\\
& -4x^6\left(3x^2-3x+1\right)\left(1107x^4-3093x^3+1863x^2-565x-28\right)z^5+6x^7\left(405x^6+5178x^5-4335x^4-128x^3+1619x^2-666x+111\right)z^6\\
& +20x^9\left(405x^4+1053x^3-1335x^2+597x-76\right)z^7+10x^{10}\left(783x^3-57x^2-69x+23\right)z^8+240x^{12}(12x-1)z^9+600x^{13}z^{10}
\end{aligned}$$

An ugly D-finite function, but nice moments

Let $m_r(n)$ be the r-th factorial moment of the distribution of black balls after n steps, i.e.

$$m_r(n) := \mathbb{E}\left(B_n(B_n-1)\cdots(B_n-r+1)\right)$$

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&= [z^n]\partial_x^r \widetilde{H}(x,z)_{|x=1} \\
&= \frac{[z^n]\left.\frac{\partial^r}{\partial x^r} H(x,z)\right|_{x=1}}{[z^n]H(1,z)}
\end{aligned}
$$

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Theorem

The r-th factorial moment satisfies

$$m_r(n) = \frac{3^r}{2^{2r/3}} \frac{\Gamma\left(\frac{r}{3}+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} n^{2r/3} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

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In particular

$$\lim_{n\to\infty} \frac{2^{2r/3}}{3^r}\frac{m_r(n)}{n^{2r/3}} = \frac{\Gamma\left(\frac{r}{3}+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} =: m_r.$$

Mittag-Leffler and generalized gamma distributions

Torsten Carleman
(1892-1949)

Gösta Mittag-Leffler
(1846-1927)

Theorem

The distribution of Young–Pólya urn is characterized by its moments.

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Proof.

[Carleman 1923] & [Fréchet, Shohat 1930] $\Rightarrow$ the moments determine the distribution uniquely.

$\square$

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Proof.

[Carleman 1923] & [Fréchet, Shohat 1930] $\Rightarrow$ the moments determine the distribution uniquely.
The support of the distribution plays a role. There is a unique distribution with such moments if the Carleman's condition holds:
– for support $[0, \infty)$ (Stieljes moment problem): if $\sum 1/m_r^{1/2r} = \infty$
– for support $(-\infty, \infty)$ (Hamburger moment problem): if $\sum 1/m_{2r}^{1/2r} = \infty$ $\square$

Young–Pólya urn of period p and parameter ℓ

The same approach allows us to study the distribution of black balls for the urn with replacement matrices

$$M_1 = M_2 = \cdots = M_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$M_p = \begin{pmatrix} 1 & \ell \\ 0 & 1+\ell \end{pmatrix}.$$

We call this model the *Young–Pólya urn of period p and parameter ℓ*.

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We call this model the *Young–Pólya urn of period p and parameter ℓ.*

Theorem

Let b_0 be the initial black and w_0 the initial white balls. Then,

$$\frac{p^\delta}{p+\ell}\frac{B_n}{n^\delta} \quad \xrightarrow{\mathcal{L}} \quad \mathrm{Beta}(b_0, w_0)\prod_{i=0}^{\ell-1}\mathrm{GenGamma}(b_0+w_0+p+i, p+\ell),$$

with $\delta = p/(p+\ell)$, and where $\mathrm{Beta}(b_0, w_0)$ is the law with support $[0,1]$ and density $\frac{\Gamma(b_0+w_0)}{\Gamma(b_0)\Gamma(w_0)}x^{b_0-1}(1-x)^{w_0-1}$.

Young Tableaux

Limiting objects

Classical dream: universality via limiting objects for combinatorial stuctures:

- Dyck paths $\rightsquigarrow$ Brownian motion (Bachelier, Einstein)
- Trees $\rightsquigarrow$ Continuous random trees (Aldous)
- Domino tilings $\rightsquigarrow$ Gaussian Free Field (Kenyon)
- Planar maps $\rightsquigarrow$ Brownian map (Marckert, Mokkadem: existence, Le Gall: triangulations, Miermont: quadrangulations)
- Self-avoiding processes $\rightsquigarrow$ SLE (Schramm, Lawler, Werner)
- Young tableaux $\rightsquigarrow$?
  - Fluctuations in the corners of rectangular shapes: Gaussian
  - Fluctuations along the edge of square shapes: Tracy–Widom limit law

Triangular Young tableaux

Triangular Young tableaux

A triangular Young tableau of slope $\alpha := -\frac{\ell}{p}$ and of size N is a classical Young tableau with N cells such that

- the first p rows (from the bottom) have length $n\ell$,
- the next p rows have length $(n-1)\ell$ and so on

6		
2	4	
1	3	5

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We are interested in $n \to \infty$.
In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 3$

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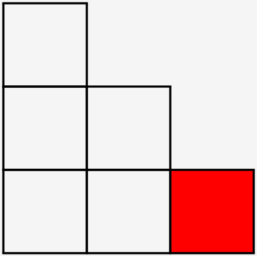

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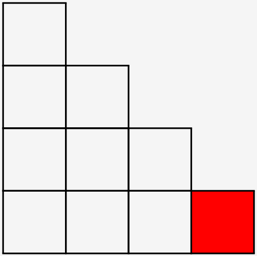

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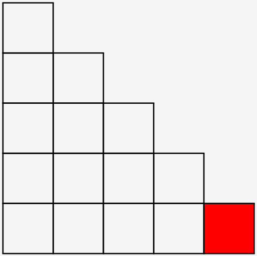

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- $p = 1, \ell = 1$
- $n = 5$

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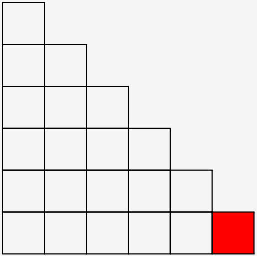

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Here

- $p = 1, \ell = 1$
- $n = 6$

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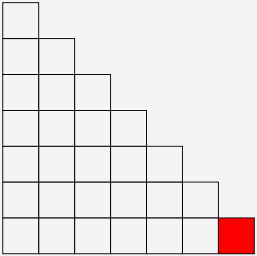

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Here

- $p = 1, \ell = 1$
- $n = 7$

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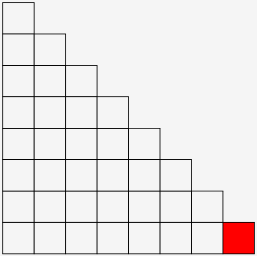

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Here

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- $n = 8$

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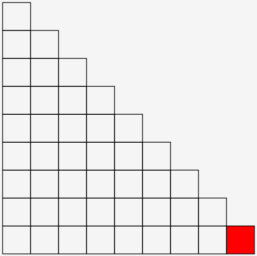

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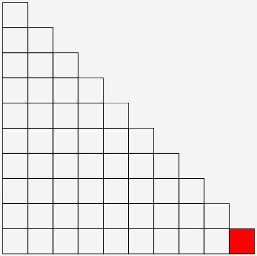

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Here

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1	2	4	9	11	16	23	33	34	36	37	42

p p p

ℓ ℓ ℓ

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Here

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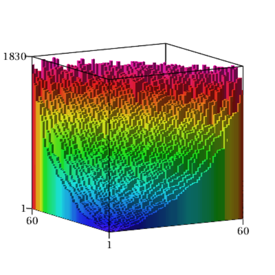
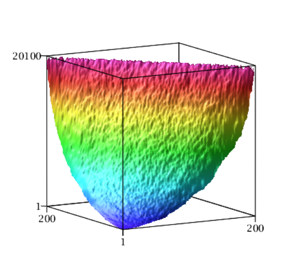
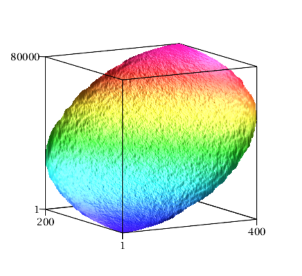
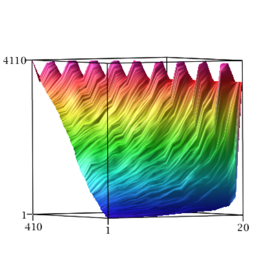
- $n = 3$

Random Young Tableaux as Random Surfaces









Asymptotics of Triangular Young Tableaux

Asymptotic results for Young Tableaux: [Logan, Shepp 77], [Vershik, Kerov 77], [Cohn, Larsen, Propp 98], [Borodin, Okounkov, Olshanski 99], [Widom 01], [Okounkov, Reshetikhin 01], [Pittel, Romik 04], [Romik 15], [Marchal 15], [Morales, Pak, Panova 16], . . .

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Theorem

Let X_n be the entry of the lower right corner of a uniform random triangular Young tableau of slope $\alpha = -\frac{\ell}{p}$ and of size N. Define $\delta = \frac{p}{p+\ell} = \frac{1}{1-\alpha}$.

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$$C\frac{N - X_n}{n^{1+\delta}}$$

converges in law to the same limiting distribution as the number of black balls in a periodic Young–Pólya urn with initial conditions $w_0 = \ell$, $b_0 = p$ and with replacement matrices $M_1 = \cdots = M_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_p = \begin{pmatrix} 1 & \ell \\ 0 & 1+\ell \end{pmatrix}$,

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$$\frac{2}{p\ell}\frac{N - X_n}{n^{1+\delta}} \xrightarrow{\mathcal{L}} \mathrm{Beta}(b_0, w_0) \prod_{i=0}^{\ell-1} \mathrm{GenGamma}(b_0 + w_0 + p + i, p + \ell).$$

Proof: **Young tableaux – Trees** – Periodic Pólya urns

Correspondence 1

41	55	61	72									
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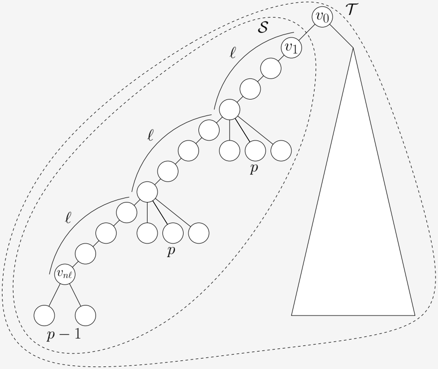
ℓ				ℓ				ℓ				



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- Lower right cell of $\mathcal{Y}$ corresponds to node $v_{n\ell}$

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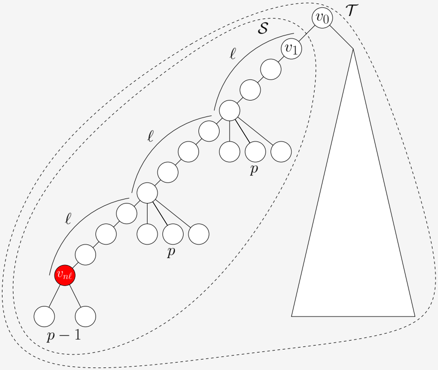
	ℓ				ℓ				ℓ			



Proof: **Young tableaux – Trees** – Periodic Pólya urns

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- First row corresponds to left-most branch of $\mathcal{S}$

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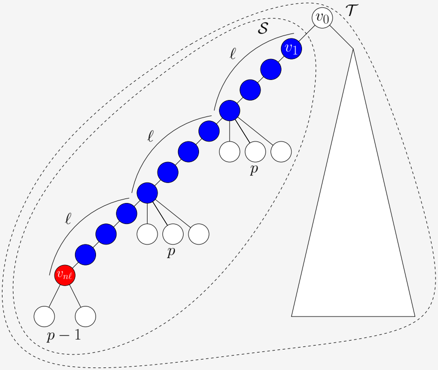

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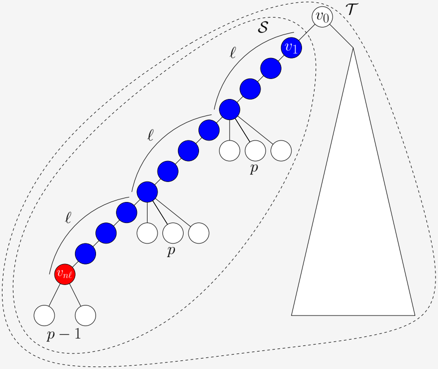
p p p

ℓ ℓ ℓ



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- First row corresponds to left-most branch of $\mathcal{S}$
- Important: Hook lengths are the same!

Key result

Let $E_\mathcal{T}$ be a uniform random linear extension of $\mathcal{T}$, and X_n be the lower right entry of $\mathcal{Y}$.

$$1 + X_n \stackrel{\mathcal{L}}{=} E_\mathcal{T}(v_{n\ell}).$$

41	55	61	72								
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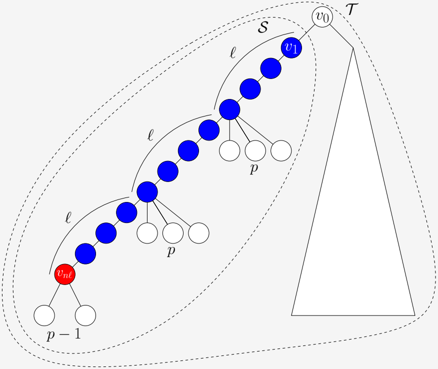
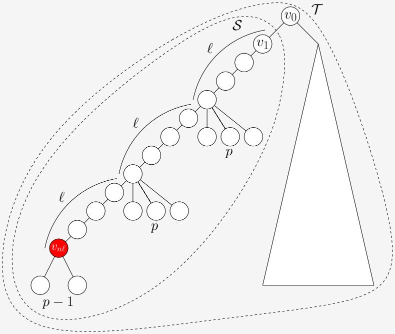
p p p

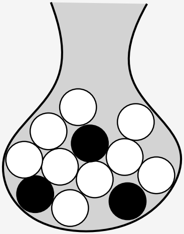
ℓ ℓ ℓ

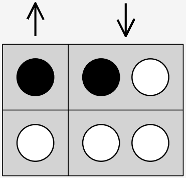


Proof: Young tableaux – **Trees** – **Periodic Pólya urns**

Correspondence 2





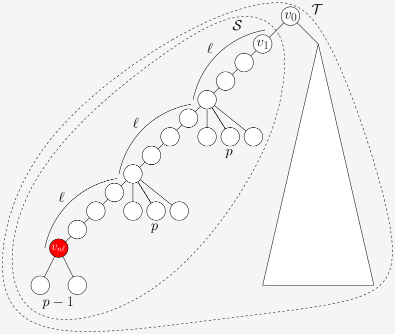


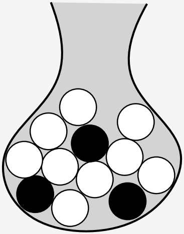
Proof: Young tableaux – **Trees** – **Periodic Pólya urns**

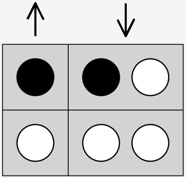
Correspondence 2

- Periodic Young–Pólya urn with period p and parameter ℓ with replacement matrices

$$M_1 = M_2 = \cdots = M_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M_p = \begin{pmatrix} 1 & \ell \\ 0 & 1+\ell \end{pmatrix}.$$







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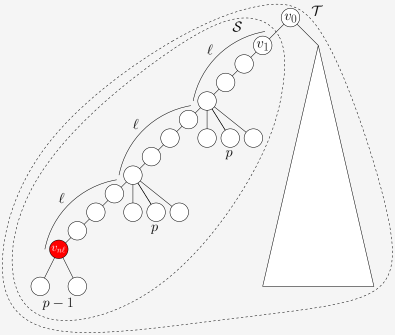

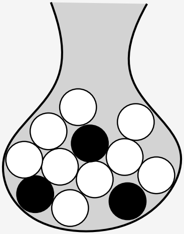

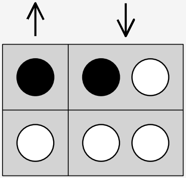

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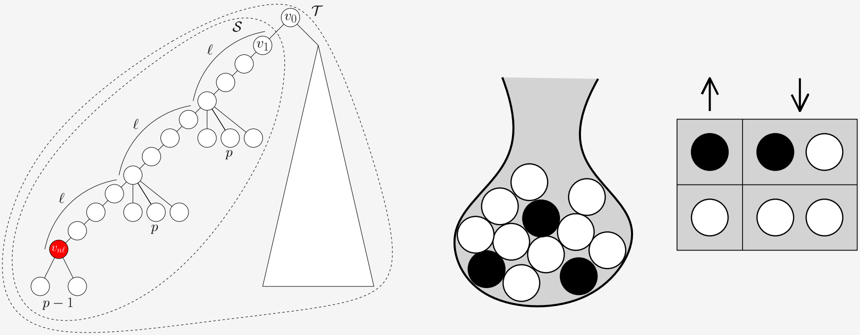
$\Rightarrow$ $N - E_{\mathcal{S}}(v_{n\ell})$ is distributed like $B_{(n-1)p}$, the black balls after $(n-1)p$ steps



The general result for Pólya urns

Theorem (The product generalized gamma distribution for balanced periodic triangular urns)

Let $p \geq 1$ and $\ell_1, \dots, \ell_p \geq 0$ be non-negative integers. Consider a periodic Pólya urn of period p with replacement matrices $M_1, \dots, M_p$ given by

$$M_j := \begin{pmatrix} 1 & \ell_j \\ 0 & 1 + \ell_j \end{pmatrix}.$$

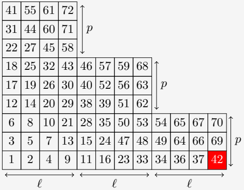
Then, the renormalized distribution of black balls is asymptotically for $n \to \infty$ given by the following product of independent distributions:

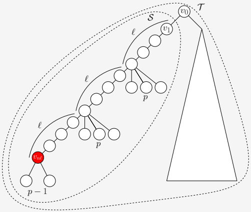
$$\frac{p^\delta}{p+\ell} \frac{B_n}{n^\delta} \quad \xrightarrow{\mathcal{L}} \quad \mathrm{Beta}(b_0, w_0) \quad \prod_{\substack{i=1 \\ i \neq \ell_1 + \cdots + \ell_j + j \text{ with } 1 \leq j \leq p-1}}^{p+\ell-1} \quad \mathrm{GenGamma}(b_0 + w_0 + i, p + \ell).$$

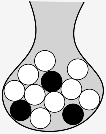
with $\ell = \ell_1 + \cdots + \ell_p$, $\delta = p/(p+\ell)$, and $\mathrm{Beta}(b_0, w_0) = 1$ when $w_0 = 0$.

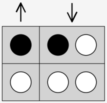
Solves the law of the south-east corner for a periodic triangular Young tableaux of any periodic shape.

Conclusion

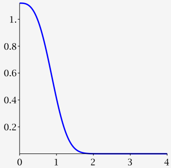






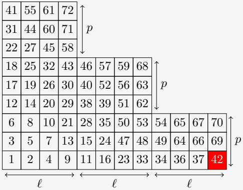


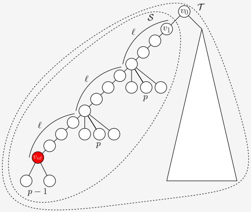
1. Extension: Periodic Pólya urns (time dependent extension)
2. Main result: Asymptotic analysis of composition
3. New limit law: Generalized Gamma distribution
4. Application: Lower right corner of triangular Young tableaux

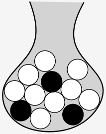


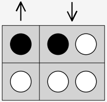
$$\mathrm{GenGamma}(1,3): \quad \frac{3\,e^{-x^3}}{\Gamma(1/3)}$$

Conclusion









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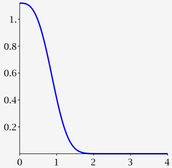

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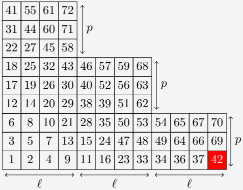

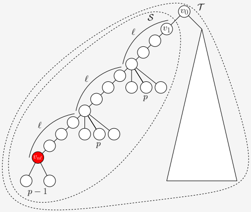

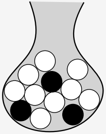

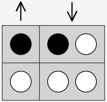

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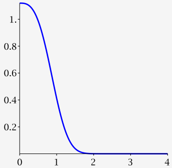

GenGamma$(1,3)$: $\frac{3\,e^{-x^3}}{\Gamma(1/3)}$

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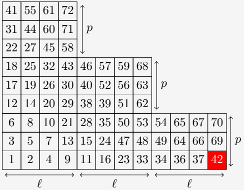

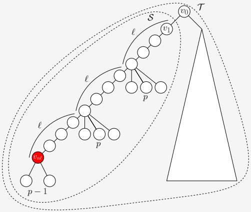

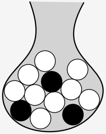

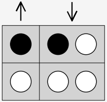

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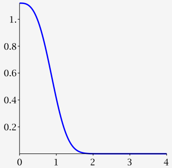

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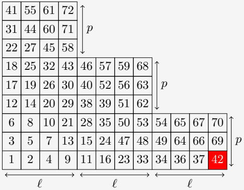

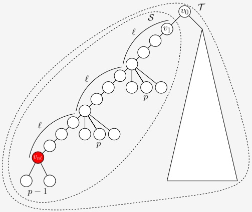

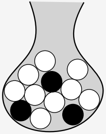

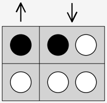

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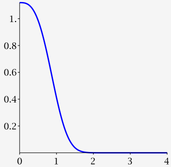

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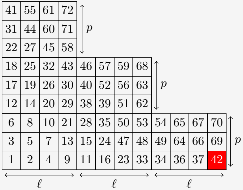

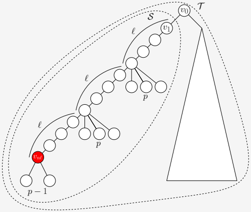

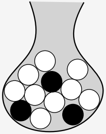

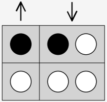

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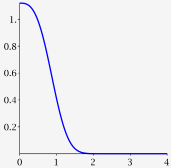
Merci !



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Thanks!