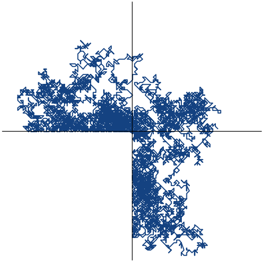
Journées de Combinatoire de Bordeaux 2019

Marches dans trois quarts de plan

Amélie Trotignon

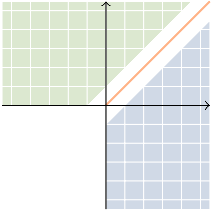
- Supervised by Marni Mishna and Kilian Raschel -

Simon Fraser University - Université de Tours

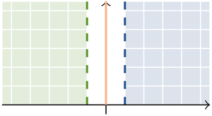


How can we enumerate walks in the three quarter plan?

I. Partition the walks by their endpoints



II. Transform the cone



III. Integral expression of the generating function

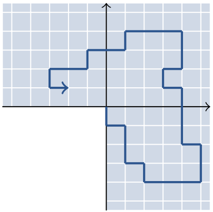
Content

Content

Introduction to walk models

A walk is a sequence of jumps from a step set $\mathcal{S}$.

We are interested in walks that start at the origin, **take steps in** $\mathcal{S}$ and remain in **the three quarter plane**.

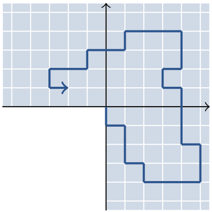


$$\mathcal{S} = \{(1,0),(0,1),(-1,0),(0,-1)\}$$

Introduction to walk models

A walk is a sequence of jumps from a step set $\mathcal{S}$.

We are interested in walks that start at the origin, **take steps in** $\mathcal{S}$ and remain in **the three quarter plane**.



$$\mathcal{S} = \{(1,0),(0,1),(-1,0),(0,-1)\}$$

Objective

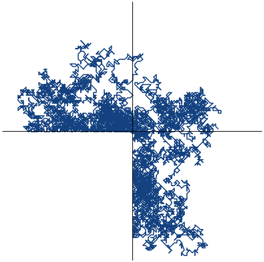
The goal is to **compute** the **number of paths** $c_{i,j}(n)$ of length n, starting at $(0,0)$ and ending at (i,j), with ($i \geq 0$ **or** $j \geq 0$) and $n \geq 0$.

Walks in the three quarter plane

We focus on walks with small steps: $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0,0)\})$

We study the generating function of walks with small steps

$$C(x,y) = \sum_{n\geq 0} \sum_{(i,j)\in\mathcal{C}} c_{i,j}(n) x^i y^j t^n$$



Strategy

We want to find an explicit expression of the generating function $C(x,y)$.

Context

M. Bousquet-Mélou, 2016. *Square lattice walks avoiding a quadrant*.

- Simple and Diagonal walks with various starting points
- Exact expression of the generating function
- Nature of the generating function
- New proof of Gessel's conjecture via reflection principle

S. Mustapha, 2016. *Non-D-finite excursions walks in a three-quadrant cone*.

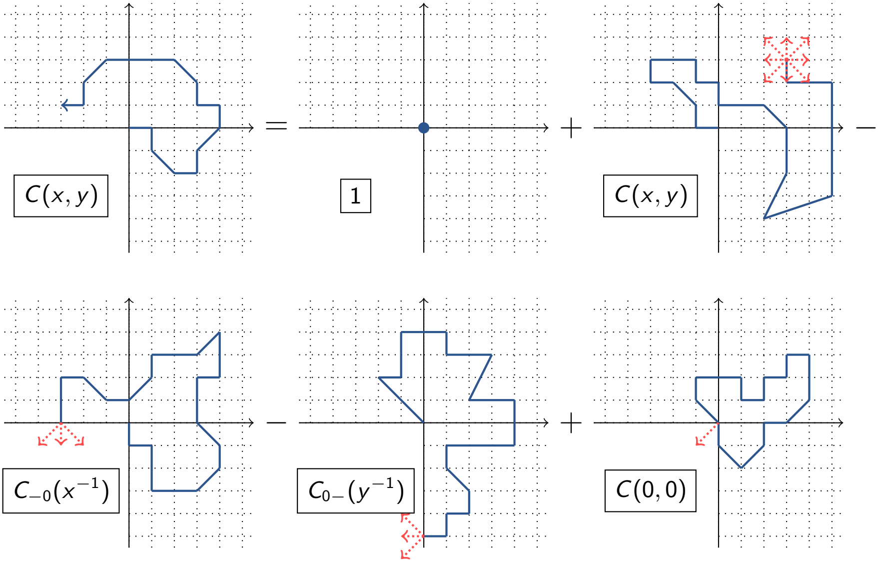
- Computation of the asymptotic of the number of excursions

T. Budd, 2017. *Winding angle of simple walks on the square lattice*.

- Enumerating formulas for planar walks keeping track of the winding angle
- Can be used to enumerate simple walks in the three quarter plane

Content

A combinatorial decomposition - $\mathcal{S} = \{\pm 1, 0\}^2 \setminus \{(0,0)\}$



A first functional Equation in the three quarter plane

The combinatorial decomposition gives a generating function equation

$$C(x,y) = 1 + tS(x,y)C(x,y) - t\left(\sum_{(i,-1)\in\mathcal{S}} x^i y^{-1}\right) C_{-0}(x^{-1})$$
$$- t\left(\sum_{(-1,j)\in\mathcal{S}} x^{-1} y^j\right) C_{0-}(y^{-1}) + t\delta_{-1,-1} C(0,0).$$

Where

$$C(x,y) = \sum_{\substack{i\geq 0 \text{ or } j\geq 0\\ n\geq 0}} c_{i,j}(n) x^i y^j t^n, \quad C_{-0}(x^{-1}) = \sum_{i\leq 0, n\geq 0} c_{i,0}(n) x^i t^n,$$
$$C_{0-}(y^{-1}) = \sum_{j\leq 0, n\geq 0} c_{0,j}(n) y^j t^n, \qquad C_{0,0} = \sum_{n\geq 0} c_{0,0}(n) t^n.$$

Compare with the functional equation in the quarter plane

$$Q(x,y) = 1 + tS(x,y)Q(x,y) - t\left(\sum_{(i,-1)\in\mathcal{S}} x^i y^{-1}\right)Q(x,0) - t\left(\sum_{(-1,j)\in\mathcal{S}} x^{-1}y^j\right)Q(0,y) + t\delta_{-1,-1}Q(0,0).$$

Where

$$Q(x,y) = \sum_{\substack{i\geq 0, j\geq 0\\ n\geq 0}} q_{i,j}(n)x^i y^j t^n, \quad Q(x,0) = \sum_{i\geq 0, n\geq 0} q_{i,0}(n)x^i t^n,$$
$$Q(0,y) = \sum_{j\geq 0, n\geq 0} q_{0,j}(n)y^j t^n, \quad Q(0,0) = \sum_{n\geq 0} q_{0,0}(n)t^n.$$

A first functional equation in the three quarter plane

$$C(x,y) = 1 + tS(x,y)C(x,y) - t\left(\sum_{(i,-1)\in\mathcal{S}} x^i y^{-1}\right) C_{-0}(x^{-1})$$
$$- t\left(\sum_{(-1,j)\in\mathcal{S}} x^{-1} y^j\right) C_{0-}(y^{-1}) + t\delta_{-1,-1} C(0,0).$$

Where

$$C(x,y) = \sum_{\substack{i\geq 0 \text{ or } j\geq 0\\ n\geq 0}} c_{i,j}(n) x^i y^j t^n, \quad C_{-0}(x^{-1}) = \sum_{i\leq 0, n\geq 0} c_{i,0}(n) x^i t^n,$$
$$C_{0-}(y^{-1}) = \sum_{j\leq 0, n\geq 0} c_{0,j}(n) y^j t^n, \qquad C_{0,0} = \sum_{n\geq 0} c_{0,0}(n) t^n.$$

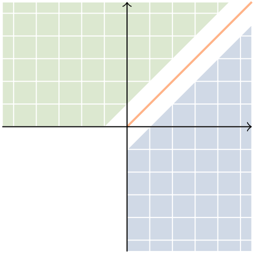
Content

Strategy

Cut the domain into three parts

We decompose the domain of possible ends of the walks into three parts:

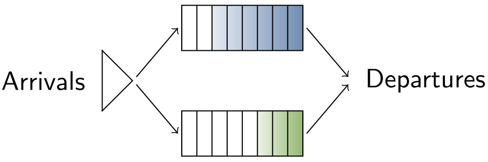
$$C(x,y) = L(x,y) + D(x,y) + U(x,y).$$

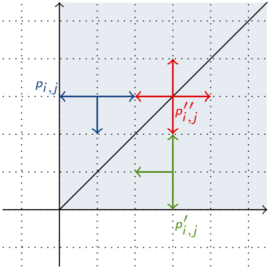


Three regions of possible endpoints.

$$\begin{cases} L(x,y) &= \displaystyle\sum_{\substack{i\geq 0\\ j\leq i-1\\ n\geq 0}} c_{i,j}(n)x^i y^j t^n, \\ D(x,y) &= \displaystyle\sum_{\substack{i\geq 0\\ n\geq 0}} c_{i,i}(n)x^i y^i t^n, \\ U(x,y) &= \displaystyle\sum_{\substack{j\geq 0\\ i\leq j-1\\ n\geq 0}} c_{i,j}(n)x^i y^j t^n. \end{cases}$$

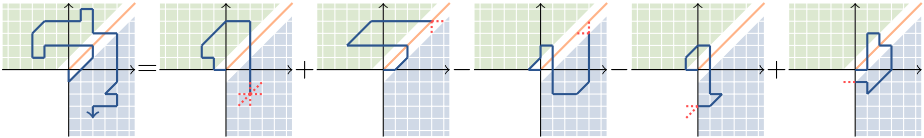
Aside: Join-the-Shortest-Queue model (JSQ)





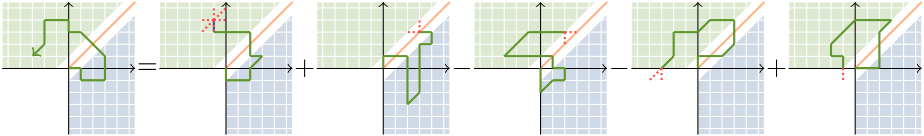
We consider a model with two queues in which the customers choose the shortest queue (if the two queues have same length, then the customers choose a queue according to a fixed probability law).

$\mathcal{S} = \{\pm 1, 0\}^2 \setminus \{(0,0), (-1,1), (1,-1)\}$



$\mathcal{S} = \{\pm 1, 0\}^2 \setminus \{(0,0),(-1,1),(1,-1)\}$

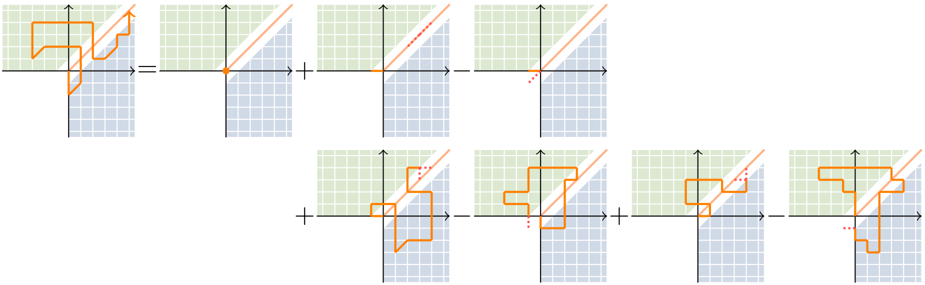
$$L(x,y) = tS(x,y)L(x,y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x,y) - t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^{\ell}(x,y)$$
$$- t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_{0-}(y^{-1}) + t\delta_{-1,0}x^{-1}\sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^{n}$$



$\mathcal{S} = \{\pm 1, 0\}^2 \setminus \{(0,0), (-1,1), (1,-1)\}$

$$L(x,y) = tS(x,y)L(x,y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x,y) - t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^{\ell}(x,y) - t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_{0-}(y^{-1}) + t\delta_{-1,0}x^{-1}\sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^n$$

$$U(x,y) = tS(x,y)U(x,y) + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D(x,y) - t(\delta_{0,-1}y^{-1} + \delta_{1,0}x)D^{u}(x,y) - t(\delta_{0,-1}y^{-1} + \delta_{-1,-1}x^{-1}y^{-1})U_{-0}(x^{-1}) + t\delta_{0,-1}y^{-1}\sum_{n\geq 0} c_{-1,0}(n)x^{-1}t^n$$



$\mathcal{S} = \{\pm 1, 0\}^2 \setminus \{(0,0), (-1,1), (1,-1)\}$

$$L(x,y) = tS(x,y)L(x,y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x,y) - t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^{\ell}(x,y) \\ - t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_{0-}(y^{-1}) + t\delta_{-1,0}x^{-1}\sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^n$$

$$U(x,y) = tS(x,y)U(x,y) + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D(x,y) - t(\delta_{0,-1}y^{-1} + \delta_{1,0}x)D^{u}(x,y) \\ - t(\delta_{0,-1}y^{-1} + \delta_{-1,-1}x^{-1}y^{-1})U_{-0}(x^{-1}) + t\delta_{0,-1}y^{-1}\sum_{n\geq 0} c_{-1,0}(n)x^{-1}t^n$$

$$D(x,y) = 1 + t(\delta_{1,1}xy + \delta_{-1,-1}x^{-1}y^{-1})D(x,y) - t\delta_{-1,-1}x^{-1}y^{-1}c_{0,0}(n) \\ + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D^{u}(x,y) - t\delta_{0,-1}y^{-1}\sum_{n\geq 0} c_{-1,0}(n)x^{-1}t^n \\ + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D^{\ell}(x,y) - t\delta_{-1,0}x^{-1}\sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^n$$

$\mathcal{S} = \{\pm 1, 0\}^2 \setminus \{(0,0), (-1,1), (1,-1)\}$

$$L(x,y) = tS(x,y)L(x,y) + t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D(x,y) - \boxed{t(\delta_{-1,0}x^{-1} + \delta_{0,1}y)D^{\ell}(x,y)}$$
$$- t(\delta_{-1,0}x^{-1} + \delta_{-1,-1}x^{-1}y^{-1})L_{0-}(y^{-1}) + \boxed{t\delta_{-1,0}x^{-1}\sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^n}$$

$$U(x,y) = tS(x,y)U(x,y) + t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D(x,y) - \boxed{t(\delta_{0,-1}y^{-1} + \delta_{1,0}x)D^{u}(x,y)}$$
$$- t(\delta_{0,-1}y^{-1} + \delta_{-1,-1}x^{-1}y^{-1})U_{-0}(x^{-1}) + \boxed{t\delta_{0,-1}y^{-1}\sum_{n\geq 0} c_{-1,0}(n)x^{-1}t^n}$$

$$D(x,y) = 1 + t(\delta_{1,1}xy + \delta_{-1,-1}x^{-1}y^{-1})D(x,y) - t\delta_{-1,-1}x^{-1}y^{-1}c_{0,0}(n)$$
$$+ \boxed{t(\delta_{1,0}x + \delta_{0,-1}y^{-1})D^{u}(x,y)} - \boxed{t\delta_{0,-1}y^{-1}\sum_{n\geq 0} c_{-1,0}(n)x^{-1}t^n}$$
$$+ \boxed{t(\delta_{0,1}y + \delta_{-1,0}x^{-1})D^{\ell}(x,y)} - \boxed{t\delta_{-1,0}x^{-1}\sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^n}$$

A system of functional equations

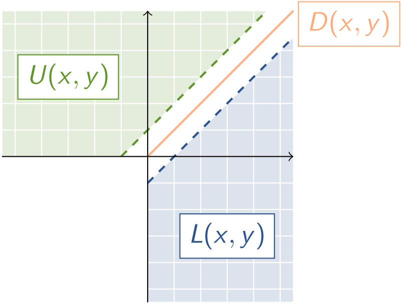
For any step set with no jumps $(-1,1)$ or $(1,-1)$ and start at $(0,0)$, one has:

$$\begin{aligned} L(x,y)K(x,y) = &- xy - \left[t\left(\delta_{-1,-1} + \delta_{1,0}x^2y + \delta_{0,-1}x + \delta_{1,1}x^2y^2\right) - xy\right] D(x,y) \\ &+ t\left(\delta_{-1,-1} + \delta_{-1,0}y\right) L_{0-}(y) - t\left(\delta_{0,-1}x + \delta_{1,0}x^2y\right) D^u(x,y) \\ &+ t\delta_{0,-1}x \sum_{n\geq 0} c_{-1,0}(n)x^{-1}t^n + t\delta_{-1,-1}c_{0,0}(n). \end{aligned}$$

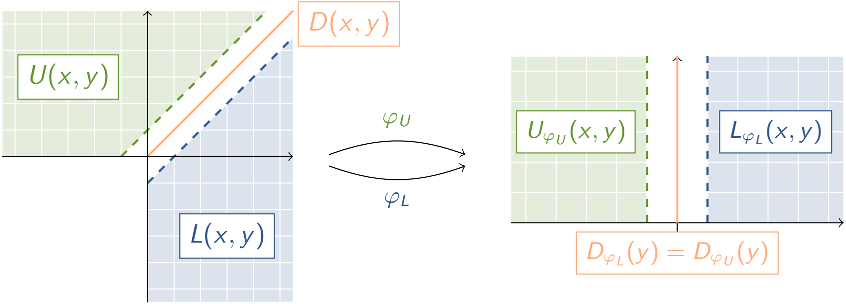
$$\begin{aligned} U(x,y)K(x,y) = &- xy - \left[t\left(\delta_{-1,-1} + \delta_{0,1}xy^2 + \delta_{-1,0}y + \delta_{1,1}x^2y^2\right) - xy\right] D(x,y) \\ &+ t\left(\delta_{-1,-1} + \delta_{0,-1}x\right) U_{-0}(x) - t\left(\delta_{-1,0}y + \delta_{0,1}xy^2\right) D^\ell(x,y) \\ &+ t\delta_{-1,0}y \sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^n + t\delta_{-1,-1}c_{0,0}(n). \end{aligned}$$

Content

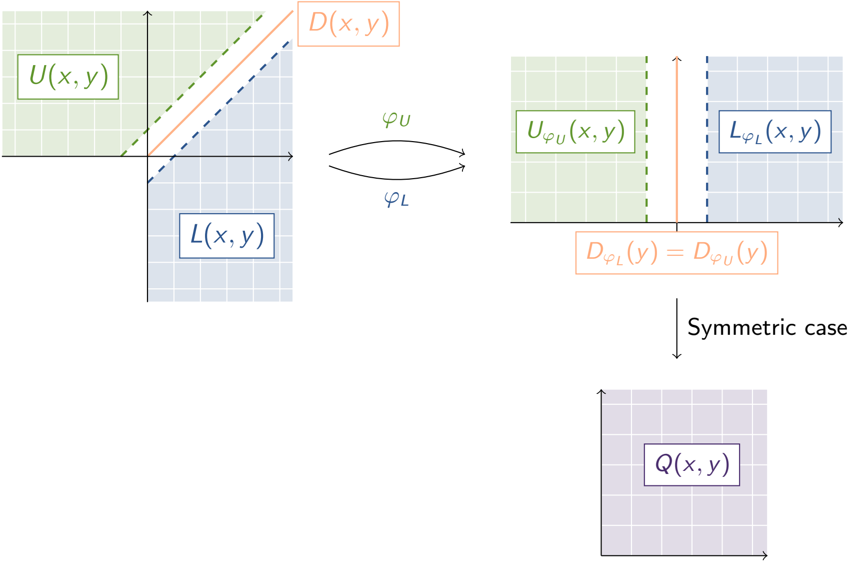
Transforming the cones



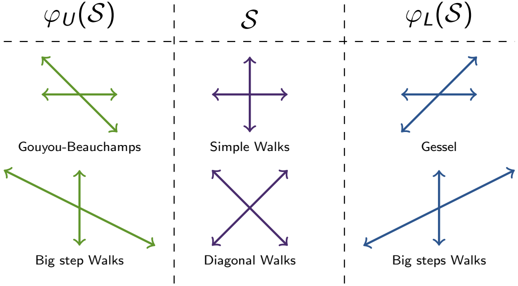
Transforming the cones



Transforming the cones



Change of variables



$$\varphi_L(x, y) = (xy, x^{-1})$$

$$\varphi_U(x, y) = (x, x^{-1}y)$$

$$\begin{cases} U_{\varphi_U}(x,y) & = & U(\varphi_U(x,y)); \\ D_{\varphi_U}(y) & = & D(\varphi_U(x,y)); \\ K_{\varphi_U}(x,y) & = & K(\varphi_U(x,y)). \end{cases} \qquad \begin{cases} L_{\varphi_L}(x,y) & = & L(\varphi_L(x,y)); \\ D_{\varphi_L}(y) & = & D(\varphi_L(x,y)); \\ K_{\varphi_L}(x,y) & = & K(\varphi_L(x,y)). \end{cases}$$

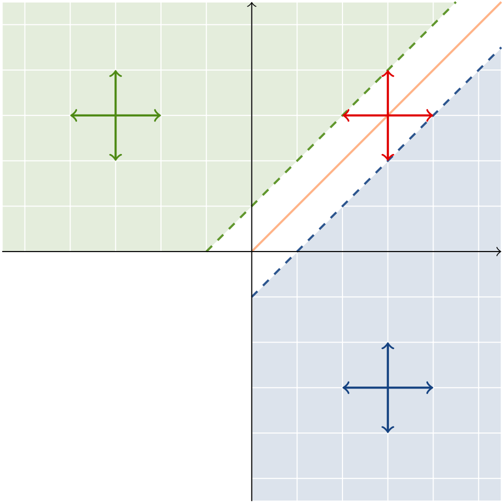
New system of functional equations

For any step set with no jumps $(-1,1)$ or $(1,-1)$ and start at $(0,0)$, one has:

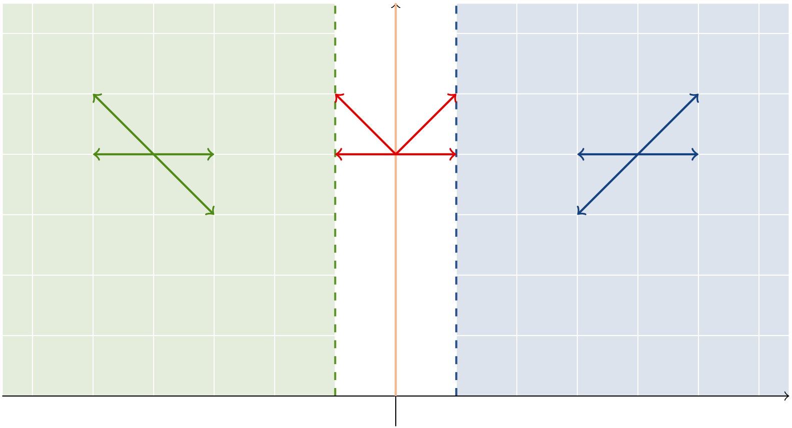
$$\begin{aligned}L_{\varphi_L}(x,y)K_{\varphi_L}(x,y) = &-xy - \left[t\left(\delta_{-1,-1}x + \delta_{1,0}x^2y^2 + \delta_{0,-1}x^2y + \delta_{1,1}xy^2\right) - xy\right] D_{\varphi_L}(y)\\ &+ t\left(\delta_{-1,-1}x + \delta_{-1,0}\right) L_{\varphi_L}(x,0) - t\left(\delta_{0,-1}x + \delta_{1,0}xy\right) D^{u}_{\varphi_U}(y)\\ &+ t\delta_{0,-1}x^2y \sum_{n\geq 0} c_{-1,0}(n)x^{-1}t^n + t\delta_{-1,-1}xc_{0,0}(n).\end{aligned}$$

$$\begin{aligned}U_{\varphi_U}(x,y)K_{\varphi_U}(x,y) = &-xy - \left[t\left(\delta_{-1,-1}x + \delta_{0,1}y^2 + \delta_{-1,0}y + \delta_{1,1}xy^2\right) - xy\right] D_{\varphi_U}(x,y)\\ &+ t\left(\delta_{-1,-1}x + \delta_{0,-1}x^2\right) U_{\varphi_U}(x) - t\left(\delta_{-1,0}x + \delta_{0,1}xy\right) D^{\ell}_{\varphi_L}(y)\\ &+ t\delta_{-1,0}y \sum_{n\geq 0} c_{0,-1}(n)y^{-1}t^n + t\delta_{-1,-1}xc_{0,0}(n).\end{aligned}$$

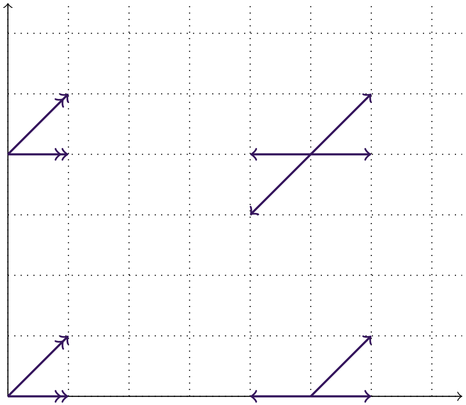
A symmetric example



A symmetric example



A symmetric example

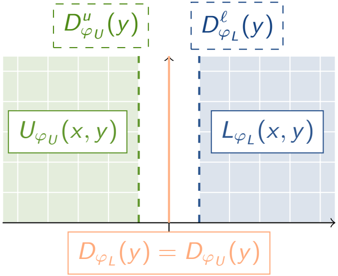


Different approach

Beaton, Owczarek and Rechnitzer, 2018. *Exact solution of some quarter plane walks with interacting boundaries*.

Content

A single functional Equation



$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

The Kernel of the Walks

$$K(x, y) = xy\left[tS(x, y) - 1\right].$$

The Kernel: a polynomial of degree 2 in x and in y

$$K(x, y) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = a(x)y^2 + b(x)y + c(x).$$

The Kernel of the Walks

$$K(x,y) = xy\left[tS(x,y) - 1\right].$$

The Kernel: a polynomial of degree 2 in x and in y

$$K(x,y) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = a(x)y^2 + b(x)y + c(x).$$

Discriminant: $\widetilde{d}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y)$ and $d(x) = b(x)^2 - 4a(x)c(x)$.

The Kernel of the Walks

$$K(x,y) = xy\,[tS(x,y) - 1]\,.$$

The Kernel: a polynomial of degree 2 in x and in y

$$K(x,y) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = a(x)y^2 + b(x)y + c(x).$$

Discriminant: $\widetilde{d}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y)$ and $d(x) = b(x)^2 - 4a(x)c(x)$.

Zeros of the Kernel, $i = 0, 1$:

$$K(X_i(y), y) = 0 \qquad\qquad K(x, Y_i(x)) = 0$$

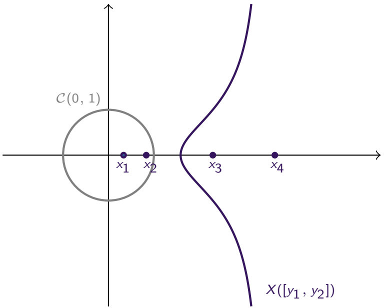
$$X_i(y) = \frac{-\widetilde{b}(y) \pm \sqrt{\widetilde{d}(y)}}{2\widetilde{a}(y)} \qquad\qquad Y_i(x) = \frac{-b(x) \pm \sqrt{d(x)}}{2a(x)}.$$

The roots of the kernel define analytic curves.

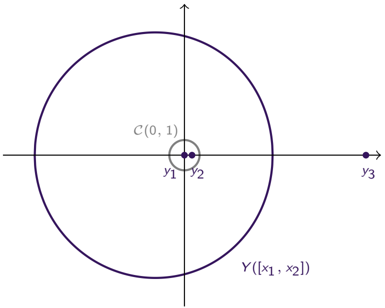
Branches of the Kernel

Kernel for Gessel's Walks

$$K(x,y) = xy\left[t\left(x + xy + x^{-1} + x^{-1}y^{-1}\right) - 1\right].$$



$X([y_1, y_2])$ for Gessel's walk ($t = 1/8$).

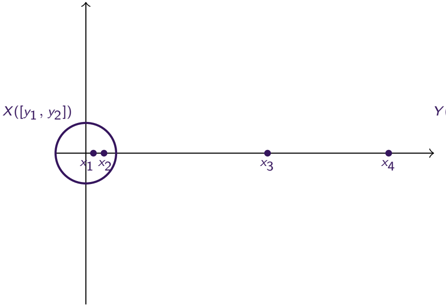


$Y([x_1, x_2])$ for Gessel's walk ($t = 1/8$).

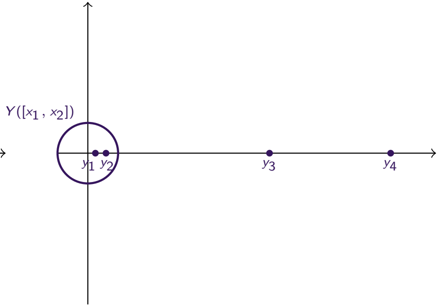
Branches of the Kernel

Kernel for Simple Walks

$$K(x, y) = xy \left[t \left(x + y + x^{-1} + y^{-1} \right) - 1 \right].$$



$X([y_1, y_2])$ for Simple walk.



$Y([x_1, x_2])$ for Simple walk.

Generating function $D(y)$ stated as a BVP

Functional Equation

$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

Generating function $D(y)$ stated as a BVP

Functional Equation

$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

Riemann-Carleman with shift BVP

By evaluating the functional equation in $Y_{\varphi_L,0}$ and $Y_{\varphi_L,1}$, we can transform the functional equation into the following **boundary value problem**:

For $y \in Y_{\varphi_L}([x_{\varphi_L,1}, x_{\varphi_L,2}])$,

$$\sqrt{\widetilde{d}_{\varphi_L}(y)}D_{\varphi_L}(y) - \sqrt{\widetilde{d}_{\varphi_L}(\bar{y})}D_{\varphi_L}(\bar{y}) = -y + \bar{y}.$$

Content

Contour-integral expression for $D(y)$

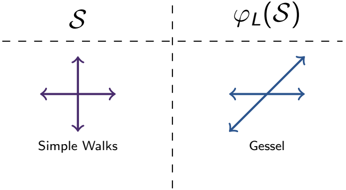
Theorem [Raschel, T., 2018]

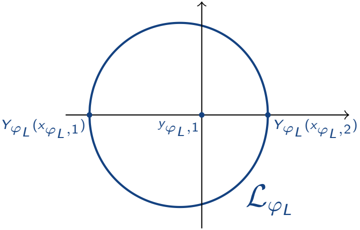
Consider a symmetric step set that does not contain the steps $(1,-1)$ and $(-1,1)$. The diagonal section have the following contour-integral expression:

$$D_{\varphi_L}(y) = \frac{-w'_{\varphi_L}(y)}{\sqrt{(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L,1})))(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L,2})))}}$$
$$\frac{1}{2i\pi} \int_{\mathcal{L}_{\varphi_L}} \frac{z w'_{\varphi_L}(z)}{\sqrt{w_{\varphi_L}(z) - w_{\varphi_L}(y_{\varphi_L,1})}(w_{\varphi_L}(z) - w_{\varphi_L}(y))} \mathrm{d}z.$$

With w_{φ_L} a known conformal mapping, and $\mathcal{L}_{\varphi_L}$ the contour $Y_{\varphi_L}([x_{\varphi_L,1}, x_{\varphi_L,2}])$.

Example of the Simple walk





w_{φ_L} (algebraic) can be expressed in terms of Weierstrass elliptic function.

Integral expression for $D_{\varphi_L}(y)$

$$D_{\varphi_L}(y) = \frac{-w'_{\varphi_L}(y)}{\sqrt{(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L,1})))(w_{\varphi_L}(y) - w_{\varphi_L}(Y_{\varphi_L}(x_{\varphi_L,2})))}} \frac{1}{2i\pi} \int_{\mathcal{L}_{\varphi_L}} \frac{z w'_{\varphi_L}(z)}{\sqrt{w_{\varphi_L}(z) - w_{\varphi_L}(y_{\varphi_L,1})}(w_{\varphi_L}(z) - w_{\varphi_L}(y))} \mathrm{d}z.$$

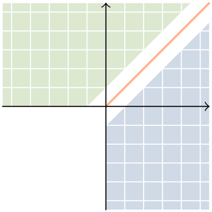
Summary

Remember - Functional Equation

$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

Remember - Domain in three parts

$$C(x,y) = L(x,y) + D(x,y) + U(x,y).$$



Symmetry of the cut and the walk

$$\Rightarrow U(x,y) = L(y,x).$$

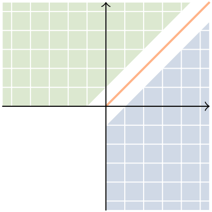
Summary

Remember - Functional Equation

$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

Remember - Domain in three parts

$$C(x,y) = {\color{blue}L(x,y)} + {\color{orange}D(x,y)} + {\color{green}L(y,x)}.$$



- We have an expression of $D_{\varphi_L}(y)$;

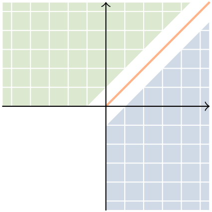
Summary

Remember - Functional Equation

$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

Remember - Domain in three parts

$$C(x,y) = L(x,y) + D(x,y) + L(y,x).$$



- We have an expression of $D_{\varphi_L}(y)$;
- With the functional equation we get an expression of $L_{\varphi_L}(x,y)$;

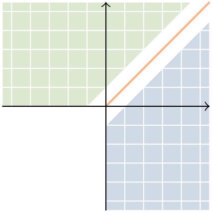
Summary

Remember - Functional Equation

$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

Remember - Domain in three parts

$$C(x,y) = L(x,y) + D(x,y) + L(y,x).$$



- We have an expression of $D_{\varphi_L}(y)$;
- With the functional equation we get an expression of $L_{\varphi_L}(x,y)$;
- With a change of variable we get an expression of $D(x,y)$ and $L(x,y)$;

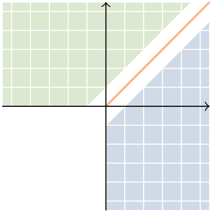
Summary

Remember - Functional Equation

$$K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) = c_{\varphi_L}(x)L_{\varphi_L}(x,0) - x\left[x\widetilde{a}_{\varphi_L}(y) + \frac{1}{2}\widetilde{b}_{\varphi_L}(y)\right]D_{\varphi_L}(y) - \frac{1}{2}xy + \frac{1}{2}tx\delta_{-1,-1}D_{\varphi_L}(0).$$

Remember - Domain in three parts

$$C(x,y) = L(x,y) + D(x,y) + L(y,x).$$



- We have an expression of $D_{\varphi_L}(y)$;
- With the functional equation we get an expression of $L_{\varphi_L}(x,y)$;
- With a change of variable we get an expression of $D(x,y)$ and $L(x,y)$;
- Then we have an expression of $C(x,y)$.

Content

Further Objectives and Perspectives

Limitations of our method

We made a restriction to:

- Symmetric step sets.
- Step sets with no jumps $(-1, 1)$ and $(1, -1)$.

Otherwise, we have to deal with too many variables and/or with walks with big steps.

Objectives

- Simplify the contour integral expression.
- Series expansion of the generating function.

References

References

- M. Bousquet-Mélou, *Square lattice walks avoiding a quadrant*, J. Combin. Theory Ser. A **144** (2016), 37–79. MR 3534064
- G. Fayolle, R. Iasnogorodski, and V. Malyshev, *Random walks in the quarter plane*, second ed., Probability Theory and Stochastic Modelling, vol. 40, Springer, Cham, 2017, Algebraic methods, boundary value problems, applications to queueing systems and analytic combinatorics. MR 3617630
- K. Raschel, *Counting walks in a quadrant: a unified approach via boundary value problems*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 3, 749–777. MR 2911883
- K. Raschel and A. Trotignon, *On walks avoiding a quadrant*, arXiv **1807.08610** (2018), 1–27.