Collapse transition of the interacting prudent walk

Nicolas Pétrélis

Joint work with Niccolò Torri

JCB, february 2019

UNIVERSITÉ DE NANTES

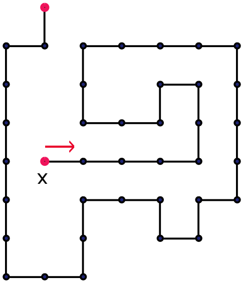
CENTRE HENRI LEBESGUE CENTRE DE MATHÉMATIQUES

1. Interacting self-avoiding walk
2. Interacting partially directed self-avoiding walk
3. Interacting prudent walk
4. Methods and Proofs

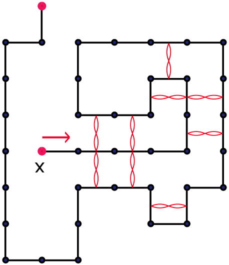
1. Interacting self-avoiding walk

1.1) Set of configurations in dimension $d \geq 2$

$$\Omega_L^{\mathrm{SAW},d} = \left\{ w = (w_i)_{i=0}^{L} \, : \, \begin{array}{l} w_0 = 0,\ w_i - w_{i-1} \in \{\pm e_1, \dots, \pm e_d\}, \\ w \text{ satisfies the self-avoiding condition} \end{array} \right\}$$



1.2) Interactions



With every $w \in \Omega_L^{\mathrm{SAW},d}$, we associate an hamiltonian that sums the self-touchings performed by w, i.e.,

$$H_L(w) = \sum_{0 \le i < j \le L} 1_{\{|u_i - u_j| = 1\}},$$

with $u_k = w_{k-1} + \frac{w_k - w_{k-1}}{2}$ center of the k-th step $(k \le L)$.

1.3) Free energy

The coupling parameter is $\beta \in [0, \infty[$ and ISAW model is then defined by

$$\mathrm{P}_{\beta,L}(w) = \frac{e^{\beta\, H_L(w)}}{Z_{\beta,L}}, \quad w \in \Omega_L^{\mathrm{SAW},d},$$

and the free energy

$$\mathrm{F}^{\mathrm{SAW}}(\beta) := \liminf_{L\to\infty} \frac{1}{L} \log Z_{\beta,L}.$$

1.4) Main open questions

- Existence of $F^{\text{SAW}}(\beta)$ for $\beta > 0$: only proven for β small [Ueltschi (2002), Hammond & Helmuth (2019)]

- Phase transition conjectured at some $\beta_c(d) > 0$ between an extended phase $\mathcal{E} = [0, \beta_c(d))$ and a collapsed phase $\mathcal{C} = [\beta_c(d), \infty)$.
  [Saleur (87), Duplantier & Saleur (87)]

- Typical extension : a typical path w sampled from $P_{\beta,L}$ is expected to scale :
  - in $\mathcal{E}$ as $||w_L|| \asymp L^{\nu_{\text{SAW}}}$,
  - in $\mathcal{C}$ as $||w_L|| \asymp L^{\chi}$ with $\chi < \nu_{\text{SAW}}$.

  [Brak, Owczarek, Prellberg (93)]

Problem : Self-avoiding walk is very complicated object ! $\Rightarrow$

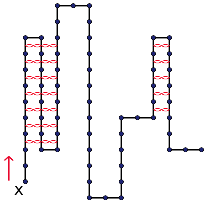
$\boxed{\text{relax the self-avoiding constraint or consider directed path.}}$

- Interacting Weakly Self-Avoiding Walk
  [v.d. Hofstadt, Klenke & Koenig (2001-2002), Bauerschmidt, Slade & Wallace (2016)]
- Interacting Partially Directed Self-Avoiding Walk
  [Zwanzig, Lauritzen (1969)]
  [Whittington, Brak, Owzcareck, Prellberg]
  [Carmona, P., Nguyen (2012, 2016, 2019)]
- Interacting Prudent Walk
  [P. & Torri (2018)]

2 Interacting partially directed self-avoiding walk

2.2) 2-dimensional Interacting partially directed self-avoiding walk [Zwanzig & Lauritzen (1968)]

$$\Omega_L^{\text{PDSAW}} = \left\{ w = (w_i)_{i=0}^L : \begin{array}{l} w_0 = 0,\ w_i - w_{i-1} \in \{\uparrow, \rightarrow, \downarrow\}, \\ w \text{ satisfies the self-avoiding condition} \end{array} \right\}$$



$$\mathbf{F}^{\text{IPDSAW}}(\beta) = \lim_{L \to \infty} \frac{1}{L} \log Z_{\beta,L}^{\text{IPDSAW}} \in [\beta, \infty).$$

2.2.1) Collapse transition

Theorem (Brack et al (1993), Nguyen & P. (2013))

Computation of $\beta_c^{\mathcal{D}} \in (0,\infty)$ *such that*

- *Collapsed phase* $\beta > \beta_c^{\mathcal{D}}$,
- *Extended phase* $\beta < \beta_c^{\mathcal{D}}$.

Phase transition second order with expo. $3/2$.

2.2.2) Path properties

The scaling limit of the path is identified in each regime.

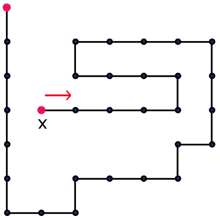
Theorem (Carmona, Nguyen & P. (2016), C. & P. (2016, 2019))

- $\beta < \beta_c^{\mathcal{D}}$: *horizontal extension of the path* $\sim L$ *and vertical extension* $\sim \sqrt{L}$
- $\beta = \beta_c^{\mathcal{D}}$: *horizontal extension of the path* $L^{2/3}$ *and vertical extension* $\sim L^{1/3}$
- $\beta > \beta_c^{\mathcal{D}}$: *Limiting Wulff shape, horizontal extension and vertical extension* $\sqrt{L}$.

3 Interacting prudent walk

3.1) Prudent paths [Debierre & Turban (1987)]

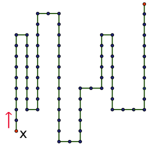
$$\Omega_L^{\mathrm{Pr}} = \left\{ w = (w_i)_{i=1}^{L} \;:\; \begin{array}{l} w_0 = 0,\; w_i \in \{\leftarrow, \uparrow, \rightarrow, \downarrow\}, \\ w \text{ satisfies the prudent condition} \end{array} \right\}$$



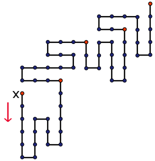
- Combinatoric viewpoint [Detheridge & Guttman (2008), Bousquet-Melou (2010), Beaton & Iliev (2015)]
- Probabilistic approach [Beffara, Friedli & Velenik (2009), P., Sun & Torri (2017)] (Scaling Limit of the Kinetic and of the Uniform prudent walk).

3.2) Families of Prudent paths

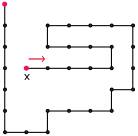
- 1-sided (partially directed)



- 2-sided (North-East)



- 4-sided (Prudent paths)



- 1-sided (partially directed)
- 2-sided (North-East)
- 4-sided (Prudent walk)

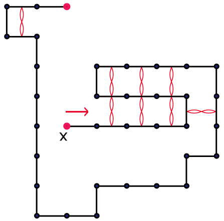
Open Question (Bousquet-Melou) : Exponential growth rate of number of configurations $\mu_{\mathtt{Pr}}$:

$$\lim_{L\to\infty} \frac{1}{L} \log \left|\Omega_L^{\mathtt{Pr}}\right| = \mu_{\mathtt{Pr}}.$$

Conjecture : $\mu_{\mathtt{Pr}} = \mu_{\mathtt{NE}}.$

3.3) Interacting Prudent Walk (IPRW)

$$\Omega_L^{\mathrm{Pr}} = \left\{ w = (w_i)_{i=1}^L \ : \ \begin{array}{l} w_0 = 0,\ w_i \in \{\leftarrow, \uparrow, \rightarrow, \downarrow\}, \\ w \text{ satisfies the prudent condition} \end{array} \right\}$$



$$\mathbf{F}^{\mathtt{Pr}}(\beta) = \lim_{L\to\infty} \frac{1}{L} \log Z_{L,\beta}^{\mathtt{Pr}} \quad \text{and} \quad \mathbf{F}^{\mathtt{NE}}(\beta) = \lim_{L\to\infty} \frac{1}{L} \log Z_{L,\beta}^{\mathtt{NE}}$$

with

$$Z_{L,\beta}^{\mathtt{Pr}} = \sum_{w\in\Omega_L^{\mathrm{Pr}}} e^{\beta H_L(w)} \quad \text{and} \quad Z_{L,\beta}^{\mathtt{NE}} = \sum_{w\in\Omega_L^{\mathrm{NE}}} e^{\beta H_L(w)}$$

3.4) Results

Theorem (P. & Torri, (2018))

1. *For any* $\beta \geq 0$ *the Free Energy exists and*

$$\mathbf{F}^{Pr}(\beta) = \mathbf{F}^{NE}(\beta) \in [\beta, \infty).$$

Consequence *(at* $\beta = 0$*) :* $\mu_P = \mu_P^{NE}$ *!*

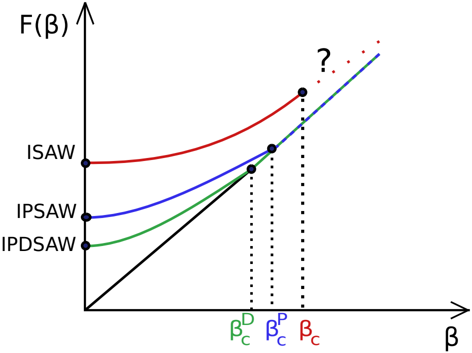
2. *There exists a critical point* $\beta_c^{Pr} \in (0, \infty)$ *such that*

–$\mathbf{F}^{Pr}(\beta) > \beta$ *for every* $\beta > \beta_c^{Pr}$,

–$\mathbf{F}^{Pr}(\beta) = \beta$ *for every* $\beta \leq \beta_c^{Pr}$.

3. $\beta_c^{Pr} \geq \beta_c^{PD}$.

4. $\mathbf{F}^{SAW}(\beta) > \beta$, *for all* $\beta \geq 0$.



F(β)
?
ISAW
IPSAW
IPDSAW
β_c^D
β_c^P
β_c
β

4 Methods and Proofs

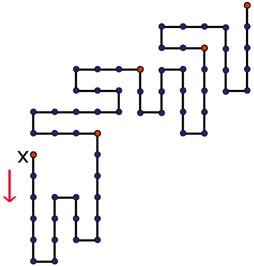
4.1) Proofs

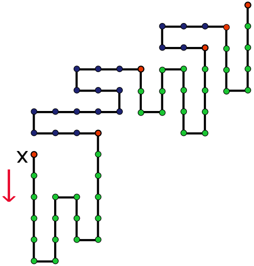
1. Existence critical point $\beta_c^{\texttt{NE}}$ such that $\mathbf{F}^{\texttt{NE}}(\beta) = \beta$, for all $\beta \geq \beta_c^{\texttt{NE}}$.
2. $\mathbf{F}^{\texttt{Pr}}(\beta) = \mathbf{F}^{\texttt{NE}}(\beta)$ for all $\beta \geq 0$.

4.2) Phase transition for the NE-model

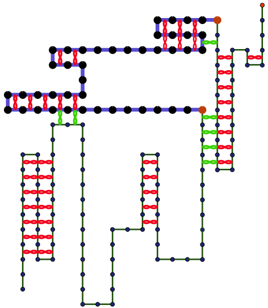
Decompose each NE path into partially directed subpaths (called blocks).

4.3) Decomposition of a path into oriented blocks




x

Critical Point : β_c : $\mathbf{F}^{\mathrm{NE}}(\beta) = \beta$ for any $\beta > \beta_c$.



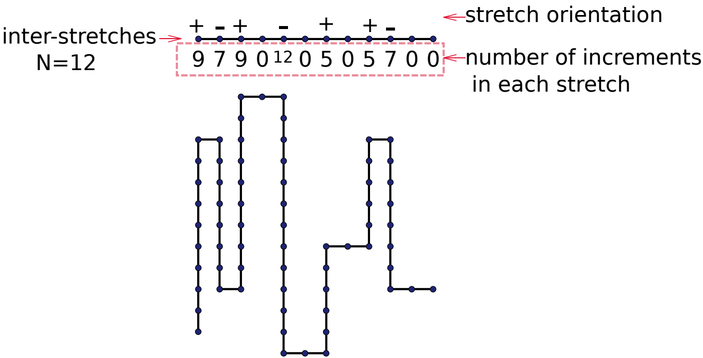
Goal : Upper bound for $\mathbf{Z}^{\mathrm{NE}}_{L,\beta}$: $\mathbf{Z}^{\mathrm{NE}}_{L,\beta} \leq C(\beta)e^{\beta L}$ for β large. Have to control

- Self-touchings in any block,
- Self-touchings between different blocks .

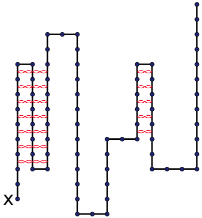
4.4) Partition function restricted to one block

4.4.1) Stretches representation of an oriented block

- $N-1$ **inter-stretches** : increments along the orientation.
- $(\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N$ sequence of **stretches**.
- Total length $|\ell_1| + \cdots + |\ell_N| + N - 1$.



$$\mathcal{L}_{T,N}(d,f) = \left\{ (\ell_i)_{i=1}^{N} :\ \ell_1 = d,\ \ell_N = f,\ \sum_{i=1}^{N} |\ell_i| = T - N + 1 \right\}$$



Number self-touchings between ℓ_i and ℓ_{i+1} :

$$\ell_i \,\tilde{\wedge}\, \ell_{i+1} := \frac{1}{2}\big(|\ell_i| + |\ell_{i+1}| - |\ell_i + \ell_{i+1}|\big).$$

$$\mathbf{Z}_{T,\beta}^{\mathrm{PD}}(N; d, f) = \sum_{\ell \in \mathcal{L}_{T,N}(d,f)} e^{\beta \sum_{i=1}^{N-1} \ell_i \,\tilde{\wedge}\, \ell_{i+1}}$$

$$= \frac{e^{\beta T - \frac{\beta}{2}|f| - \frac{\beta}{2}|d|}}{e^{\beta(N-1)}} \sum_{\ell \in \mathcal{L}_{T,N}(d,f)} \prod_{i=1}^{N-1} e^{-\frac{\beta}{2}|\ell_{i+1} + \ell_i|}$$

4.4.2) Auxiliary Random Walk

With each $(\ell_i)_{i=1}^N \in \mathcal{L}_{T,N}(d,f)$ we associate a random walk trajectory

$$(\ell_1,\dots,\ell_N) \Leftrightarrow (V_i)_{i=1}^N\colon\ V_i = (-1)^{i-1}\ell_i$$

The increments of V :

$$V_i - V_{i-1} = (-)^{i-1}(\ell_{i-1} + \ell_i).$$

One to one correspondence between $\mathcal{L}_{T,N}(d,f)$ and

$$\Big\{(V_i)_{i=1}^N\colon G_N(V) = L - N + 1,\, V_1 = d,\, V_N = (-1)^{N-1} f\Big\}$$

with $G_N(V) = \sum_{i=1}^N |V_i|$.

$$\mathbf{Z}^{\mathrm{PD}}_{T,\beta}(N;d,f) = e^{\beta T - \frac{\beta}{2}|f| - \frac{\beta}{2}|d|} \left(\frac{c_\beta}{e^\beta}\right)^{N-1} \boxed{\sum_{\ell \in \mathcal{L}_{T,N}(d,f)} \prod_{i=1}^{N-1} \frac{e^{-\frac{\beta}{2}|\ell_{i+1}+\ell_i|}}{c_\beta}}$$

with

$$\boxed{\sum_{\ell \in \mathcal{L}_{T,N}(d,f)} \dots} = \sum_{\ell \in \mathcal{L}_{T,N}(d,f)} \mathbf{P}_\beta\left((V_i)_{i=2}^N = \left((-1)^{i-1}\ell_i\right)_{i=2}^N \,\middle|\, V_1 = d\right)$$

$$\mathbf{Z}^{\mathrm{PD}}_{T,\beta}(N;d,f)$$

$$= e^{\beta T - \frac{\beta}{2}|f| - \frac{\beta}{2}|d|} \left(\frac{c_\beta}{e^\beta}\right)^{N-1} \boxed{\mathbf{P}_\beta\left(\begin{array}{c} V_N = (-1)^{N-1} f \\ G_N(V) = T - N + 1 \end{array} \,\middle|\, V_1 = d\right)}$$

4.5) Interactions between oriented blocks

Block $i+1$ interacts with blocks $i-1$ and i and such interactions are bounded above by

$$(N_i + |f_{i-1}|) \wedge |d_{i+1}| \leq \frac{|f_{i-1}| + |d_{i+1}|}{2} + \frac{3}{4}(N_i - 1) - \frac{1}{4}|f_{i-1} + d_{i+1}|$$

4.6) Full partition function

Partition Ω_L^{NE} depending on

- r number of oriented blocks.
- $T_1, \dots, T_r$ length of each block $\quad (T_1 + \dots + T_r = L)$
- $N_1, \dots, N_r$ numbers of stretches in each block
- $(d_i, f_i)_{i=1}^r$ length of first and last stretch in each block.

Thus

$$\mathbf{Z}_{L,\beta}^{\mathrm{NE}} \leq \sum_{r \leq L} \sum_{T,N,d,f} \boxed{\prod_{i=1}^{r} \mathbf{Z}_{T_i,\beta}^{\mathtt{PD}}(N_i, d_i, f_i)\, e^{\beta(|f_{i-2}|+N_{i-1}) \wedge |d_i|)}}$$

and

$$\boxed{\prod_{i=1}^{r}} \leq \prod_{i=1}^{r} e^{\beta T_i - \frac{\beta}{2}(|d_i|+|f_i|)} \left(\frac{c_\beta}{e^\beta}\right)^{N_i-1} \mathbf{P}_\beta \left(\begin{array}{c} V_{N_i} = (-1)^{N_i-1} f_i \\ G_{N_i}(V) = T_i - N_i + 1 \end{array} \,\middle|\, V_1 = d_i \right)$$

$$e^{\beta\left(\frac{|f_{i-1}|+|d_{i+1}|}{2} + \frac{3}{4}(N_i-1) - \frac{1}{4}|f_{i-1}+d_{i+1}|\right)}$$

$$\boxed{\prod_{i=1}^{r}} = e^{\beta L} \prod_{i=1}^{r} \left(\frac{c_\beta}{e^{\beta}}\right)^{N_i - 1} \mathbf{P}_\beta \left(\begin{smallmatrix} V_{N_i} = (-1)^{N_i - 1} f_i \\ G_{N_i}(V) = T_i - N_i + 1 \end{smallmatrix} \;\Big|\; V_1 = d_i \right)$$

$$\times \, e^{\beta \left(\frac{3}{4}(N_i - 1) - \frac{1}{4} |f_{i-1} + d_{i+1}| \right)}$$

so that (with $f_0 = d_{r+1} = 0$)

$$\boxed{\prod_{i=1}^{r}} = e^{\beta L} \left(\frac{c_\beta}{e^{\frac{\beta}{4}}}\right)^{N_1 + \cdots + N_r - r} c_{\beta/2}^{r}$$

$$\times \prod_{i=1}^{r} \mathbf{P}_\beta \left(\begin{smallmatrix} V_{N_i} = (-1)^{N_i - 1} f_i \\ G_{N_i}(V) = T_i - N_i + 1 \end{smallmatrix} \;\Big|\; V_1 = d_i \right) \frac{e^{-\frac{\beta}{4}|f_{i-1} + d_{i+1}|}}{c_{\beta/2}}$$

assume $r \in 2\mathbb{N}$:

$$\prod_{i=1}^{r} \mathbf{P}_\beta \left(\begin{array}{c} V_{N_i} = (-1)^{N_i - 1} f_i \\ G_{N_i}(V) = T_i - N_i + 1 \end{array} \middle| V_1 = d_i \right) \frac{e^{-\frac{\beta}{4}|f_{i-1} + d_{i+1}|}}{c_{\beta/2}}$$

$$= \prod_{i=1}^{r/2} \mathbf{P}_\beta \left(\begin{array}{c} V_{N_{2i}} = (-1)^{N_{2i} - 1} f_{2i} \\ G_{N_{2i}}(V) = T_{2i} - N_{2i} + 1 \end{array} \middle| V_1 = d_{2i} \right) \frac{e^{-\frac{\beta}{4}|f_{2i-2} + d_{2i}|}}{c_{\beta/2}}$$

$$\prod_{i=1}^{r/2} \mathbf{P}_\beta \left(\begin{array}{c} V_{N_{2i-1}} = (-1)^{N_{2i-1}} f_{2i-1} \\ G_{N_{2i-1}}(V) = T_{2i-1} - N_{2i-1} + 1 \end{array} \middle| V_1 = d_{2i-1} \right) \frac{e^{-\frac{\beta}{4}|f_{2i-1} + d_{2i+1}|}}{c_{\beta/2}}$$

Consequence : The summations over $T_1, \ldots, T_r$ and over $(d_i, f_i)_{i=1}^r$ disappear in the probabilities.

Conclusion : Since $N_i \geq 2$ for every $i \leq r$ (at least two stretches in an oriented block)

$$\mathbf{Z}^{\mathtt{NE}}_{L,\beta} \leq e^{\beta L} \sum_{r=1}^{L/4} \sum_{N_1+\cdots+N_r \leq L} c_{\beta/2}^r \left(\frac{c_\beta}{e^{\frac{\beta}{4}}}\right)^{N_1+\cdots+N_r-r}$$

$$\leq e^{\beta L} \sum_{r=1}^{\infty} \left[c_{\beta/2} \sum_{N=1}^{\infty} \left(\frac{c_\beta}{e^{\frac{\beta}{4}}}\right)^N\right]^r$$

That is, $\mathbf{Z}^{\mathtt{NE}}_{L,\beta} \leq C(\beta) e^{\beta L}$ if β is large enough. $\square$