

Une notion de récurrence dans le modèle du tas de sable sur le réseau carré

Henri Derycke
joint work with Yvan Le Borgne

LaBRI

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université
de **BORDEAUX**

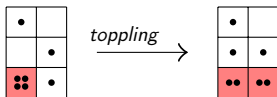
Sandpile Model [Bak, Tang, Wiesenfeld 87]



Configuration: $\eta : V \mapsto \mathbb{N}$

$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.

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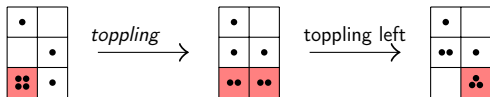


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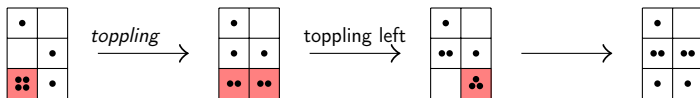


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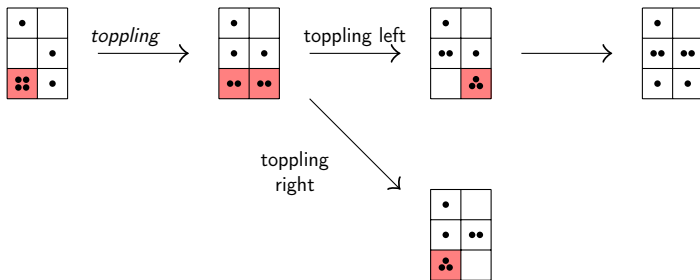


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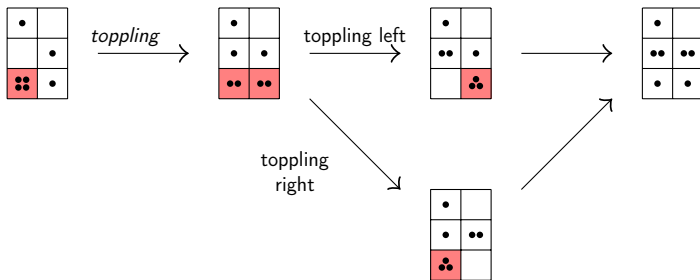


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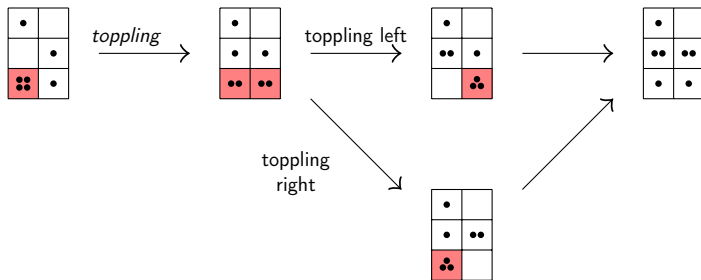


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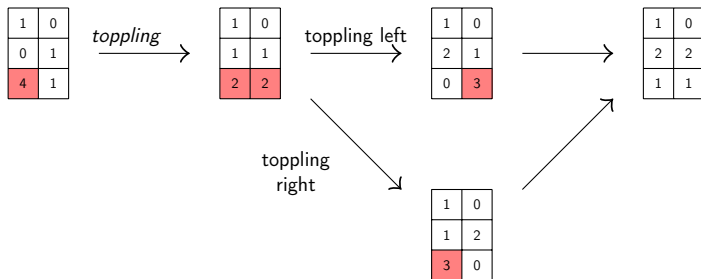
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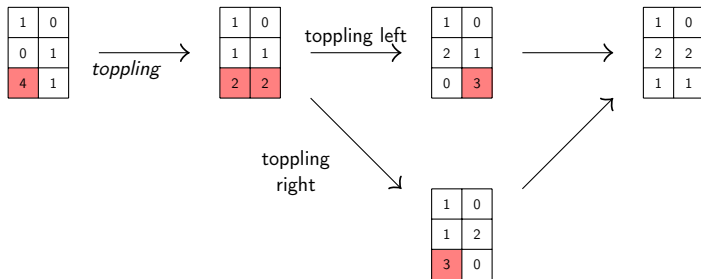
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Stabilisation



Stabilisation : while a vertex is unstable, topple it.

The sink

How to stabilize (even with a large number of grains)?

2	2
3	3
2	2

The sink

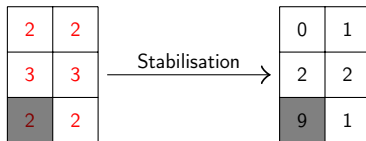
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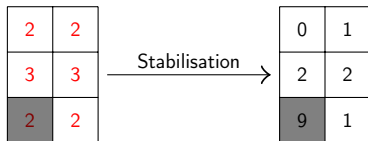


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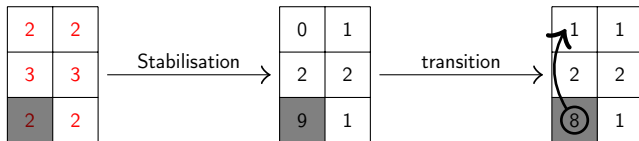
Markov Chain

- ▶ States: stable configurations on G
- ▶ Transition: Add a particle from the sink to a vertex chosen uniformly and stabilize

Recurrent states are in the same connected component.

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Dhar operator

Topple the sink (forced), then stabilize: $\text{dhar}(\eta) := \text{stab}(\eta + \Delta^{(s)})$

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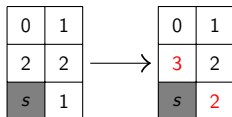
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0	1
2	2
s	1

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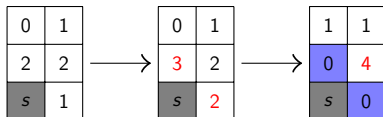
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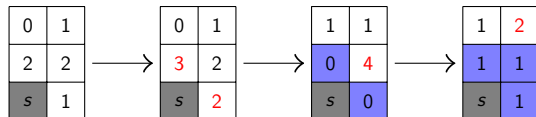
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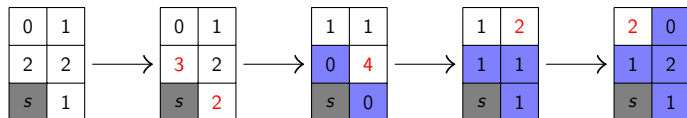
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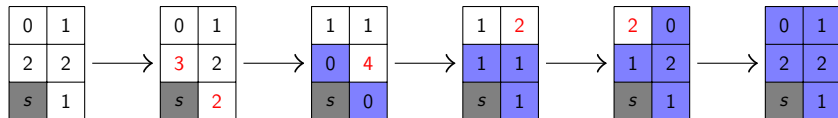
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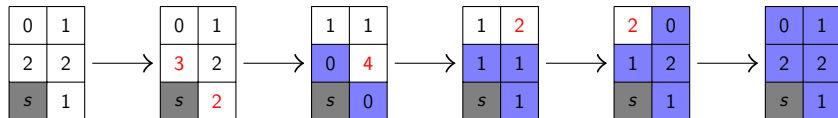
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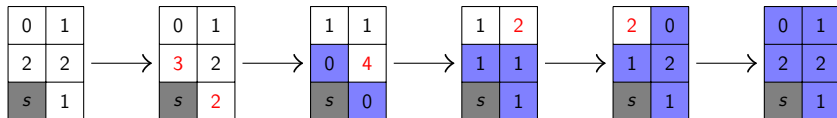
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A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.

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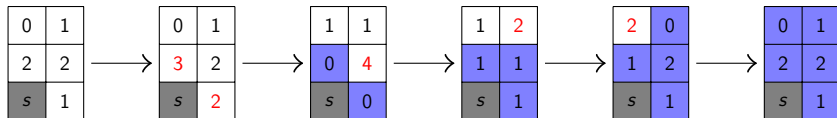
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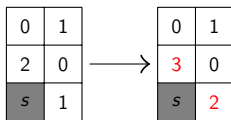
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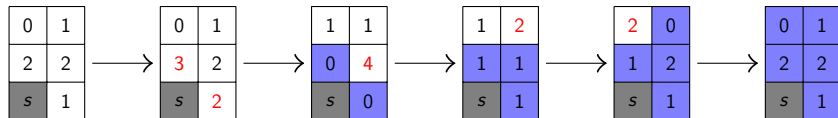
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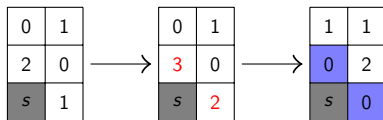
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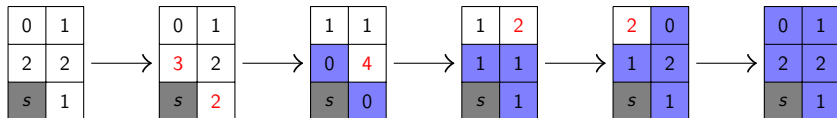
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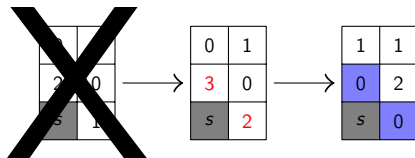
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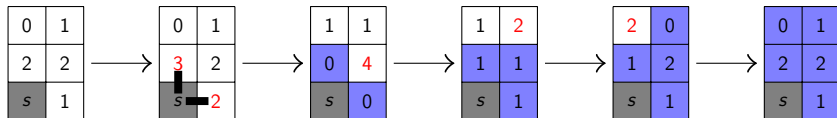
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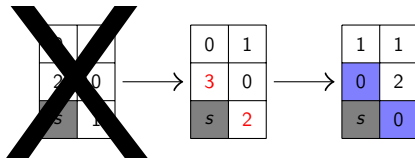
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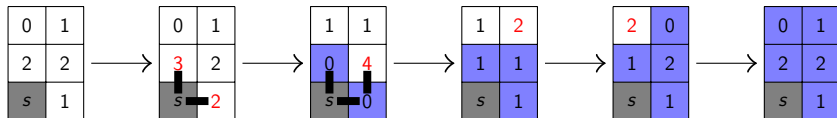
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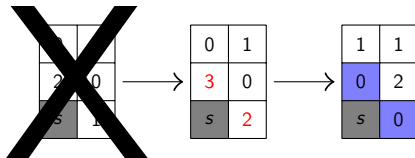
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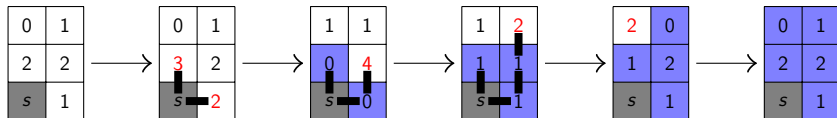
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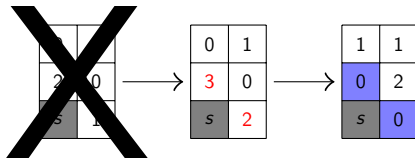
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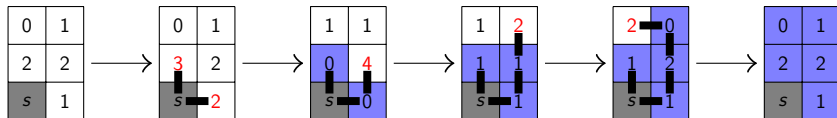
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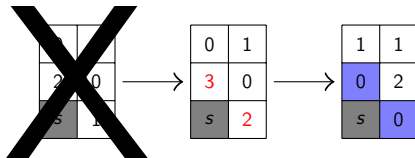
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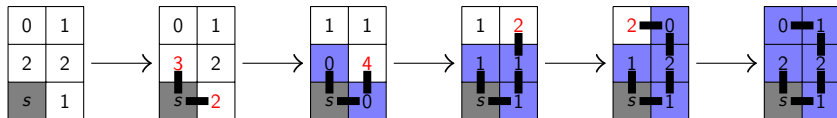
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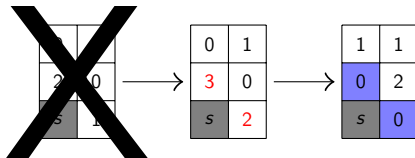
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Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

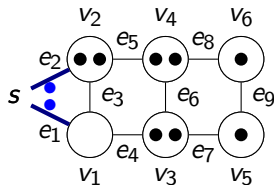
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrences on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)



Mark edges incident to the sink as pending edges.

While there is a pending edge

Get the closest pending edge to the sink

Process the grain(s) on the edge

If a vertex become unstable, topple it and mark its untreated incident edges as pending edges.

Edge-vertex traversal: s ,

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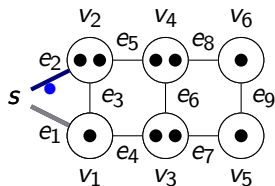
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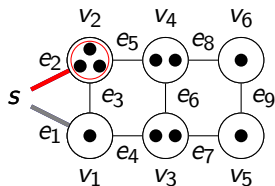
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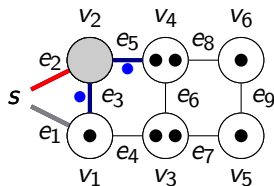
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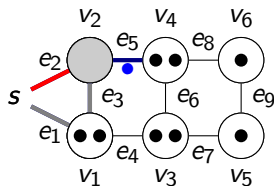
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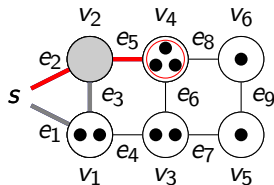
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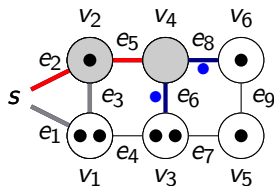
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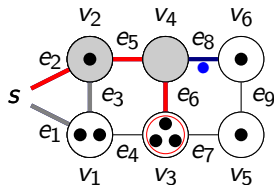
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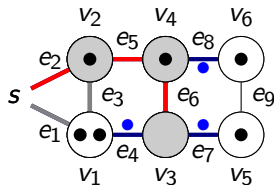
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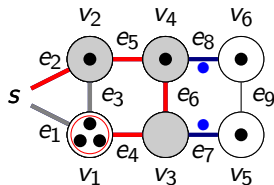
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Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

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Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)



Mark edges incident to the sink as pending edges.

While there is a pending edge

Get the closest pending edge to the sink

Process the grain(s) on the edge

If a vertex become unstable, topple it and mark its untreated incident edges as pending edges.

Edge-vertex traversal: $s, e_1, v_2, e_3, e_5, v_4, e_6, v_3, e_4,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

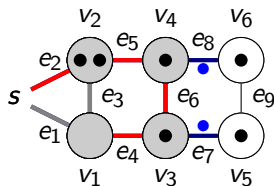
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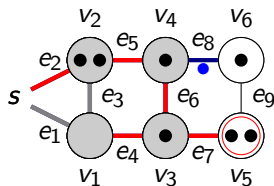
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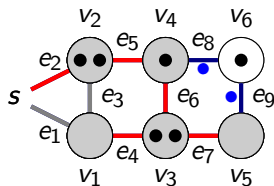
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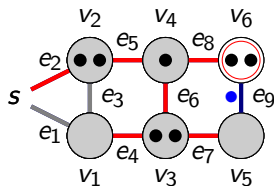
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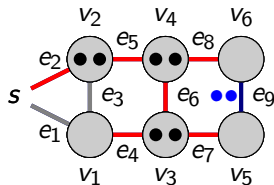
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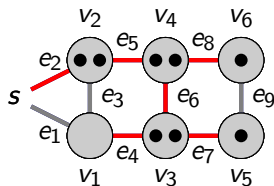
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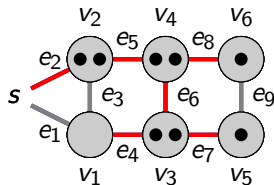
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Counting grains

In Dhar criterion, each edge captures the last grain that crosses it.
For any recurrent configuration η on $G = (V \cup \{s\}, E)$,

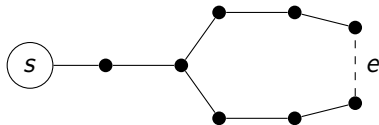
$$\text{level}(\eta) = \left(\sum_{v \in V} \eta(v) \right) + \deg(s) - |E|.$$

Let $R_G(y) = \sum_{\eta \in \text{Rec}(G,s)} y^{\text{level}(\eta)}$

Theorem (López 97)

For any graph $G = (V \cup \{s\}, E)$,

$$R_G(y) = \text{Tutte}_G(1, y).$$



where $\text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}(T)}$ counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.

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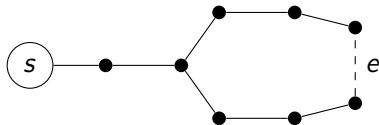
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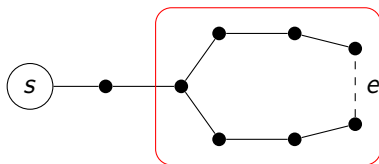
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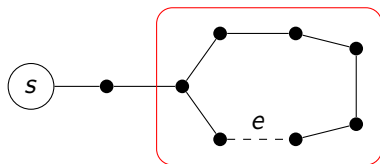
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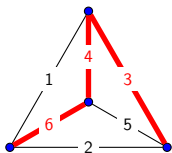
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Tracking external activity while changing order on edges

With $e_1 <_E e_2 <_E \dots <_E e_{|E|}$ an order on the edges of E , an external edge is *active* if it is maximal for $<_E$ in its fundamental cycle.

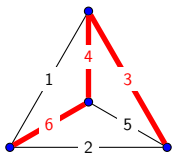


Proposition

$\text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}_{<_E}(T)}$ does not depend on $<_E$

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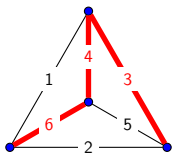
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$\{e_i, e_j\}$ is a critical pair if

- ▶ e_i is external
- ▶ e_j is on e_i fundamental cycle
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Let τ_i exchanging e_i and e_{i+1} in $<_E$.

$$\Phi_i(T) = \begin{cases} T \Delta \{e_i, e_{i+1}\} & \text{if } \{e_i, e_{i+1}\} \text{ is a critical pair of } T \\ T & \text{otherwise} \end{cases}$$

Lemma: for all T $\text{ext}_{<_E}(T) = \text{ext}_{\tau_i(<_E)}(\Phi_i(T))$.

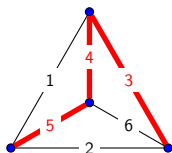
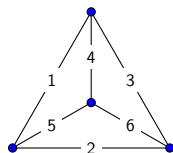
Tutte Polynomial

Let a graph $G = (V, E)$ and $<_E$ an order on the edges of E .

$$\text{Tutte}_G(x, y) = \sum_{T \in \Sigma(G)} x^{\text{int}(T)} y^{\text{ext}(T)}$$

Active external edge: maximal in its fundamental cycle.

Active internal edge: maximal in its co-cycle.



e_6 is active with fundamental cycle (e_3, e_4, e_6) . e_5 is active with co-cycle (e_1, e_2, e_5) .

For $G = K_4$, $\text{Tutte}_G(x, y) = x^3 + y^3 + 3x^2 + 4xy + 3y^2 + 2x + 2y$ and T weights xy .

When G is planar, $\text{Tutte}_G(x, y) = \text{Tutte}_{G^*}(y, x)$. Then if planar and self-dual, $\text{Tutte}_G(x, y) = \text{Tutte}_G(y, x)$

Checkpoint

Finite graphs

▷ Stable configurations

▷ Dhar Criterion

▷ Bijection between recurrent and spanning trees

▷ Tutte polynomial

▷ Invariant by edge exchange

▷ Symmetric for self-dual planar graphs

Checkpoint

Finite graphs

Square lattice (biperiodicity)

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Square lattice (biperiodicity)

▷ Biperiodic stable configurations

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Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
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▷ Tutte polynomial	
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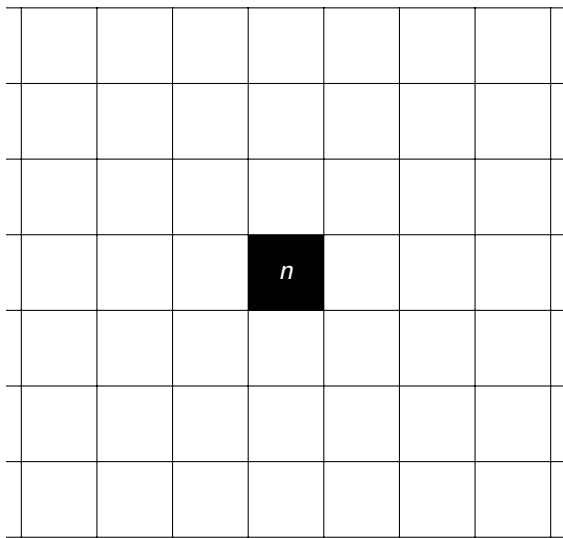
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▷ Symmetric for self-dual planar graphs	▷ Symmetric joint distribution of external/internal activities <i>changing</i> by rotation

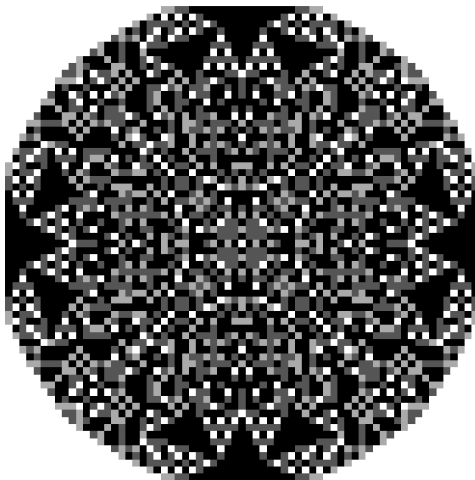
Some definition of recurrence for $\mathbb{Z}^2$

From wired uniform spanning forest [Gamlin, Jarai] with an anchor burning bijection.

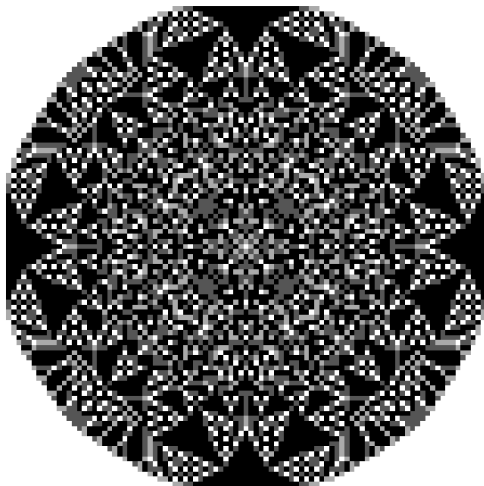
Local description in probability [Priezzhev, Ruelle]

Sandpile identity: $\lim_{n \rightarrow \infty} \text{dhar}^n(0^{\mathbb{Z}^2})$? [Paoletti, Caracciollo, Sportiello, Levine, Pegden, Smart...]

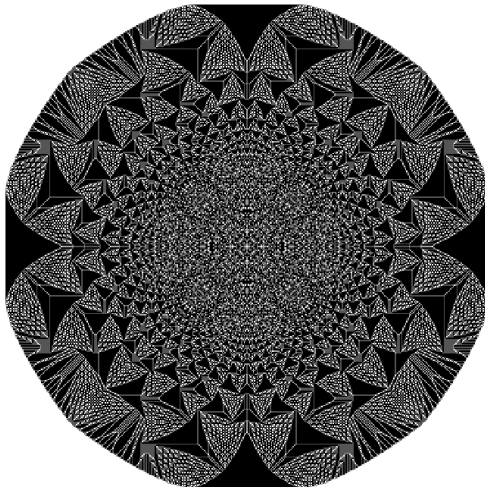




Source: W.Pegden, $n = 2^{13}$

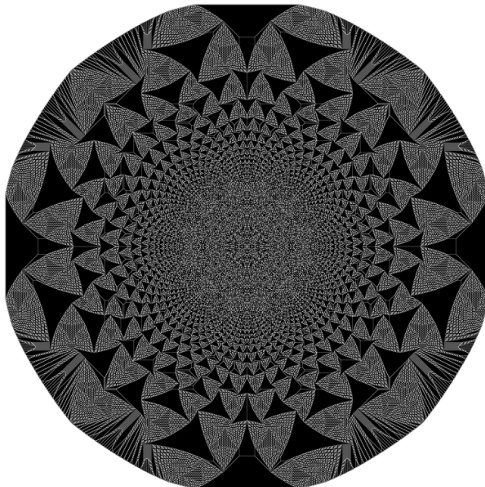


Source: W.Pegden, $n = 2^{14}$



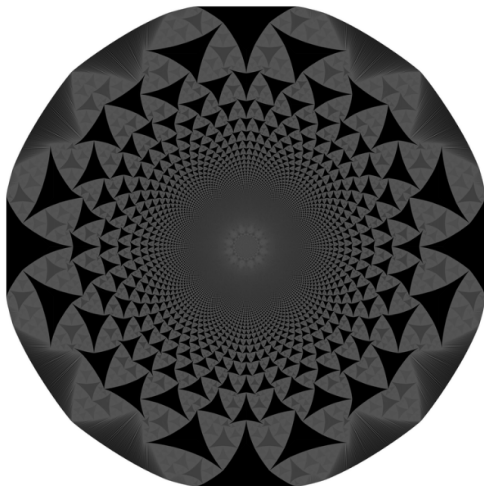
Source: W.Pegden, $n = 2^{18}$

- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]



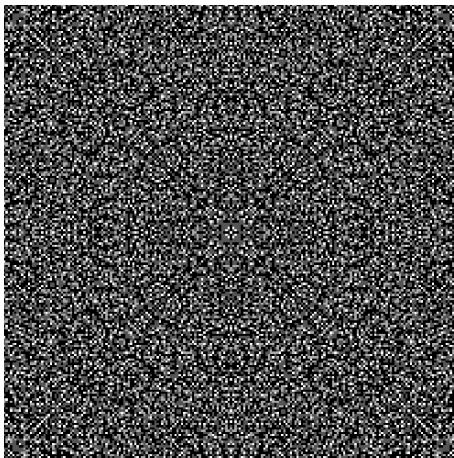
Source: W.Pegden, $n = 2^{20}$

- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]



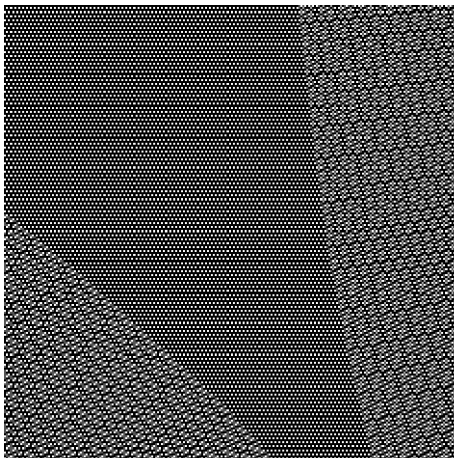
Source: W.Pegden, $n = 2^{30}$

- ▶ Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]
- ▶ Convergence in terms of density [Pegden, Smart 12]



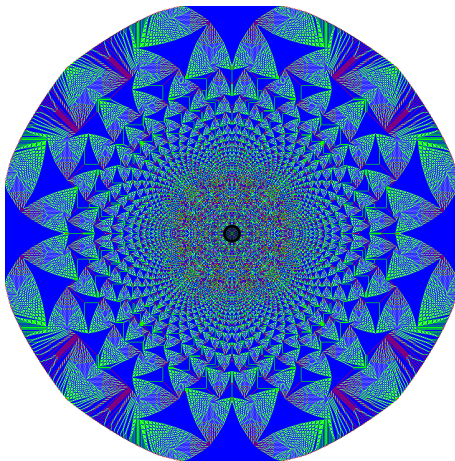
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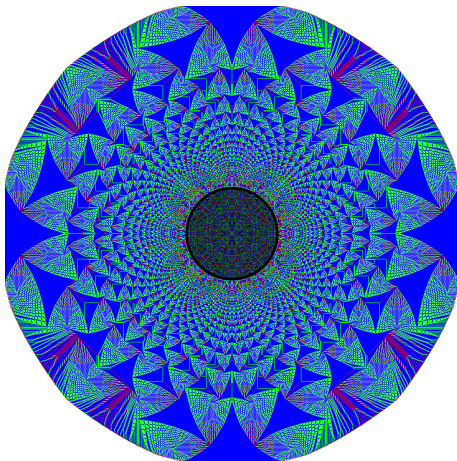
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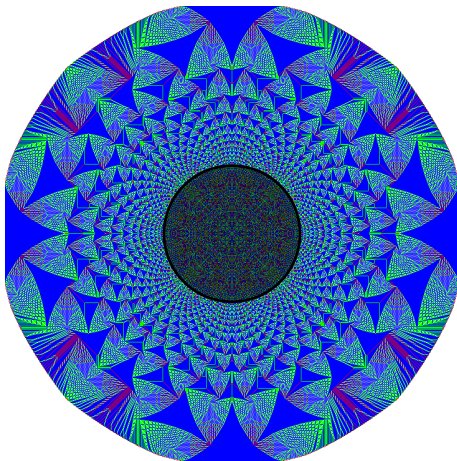
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



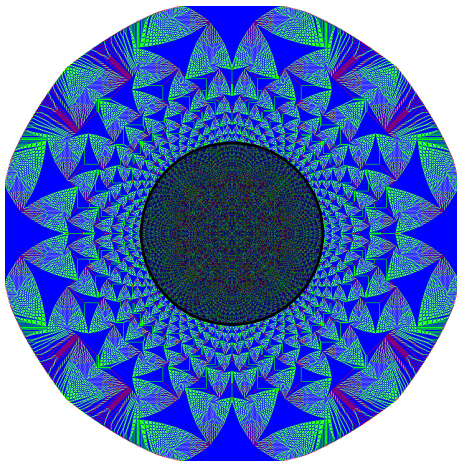
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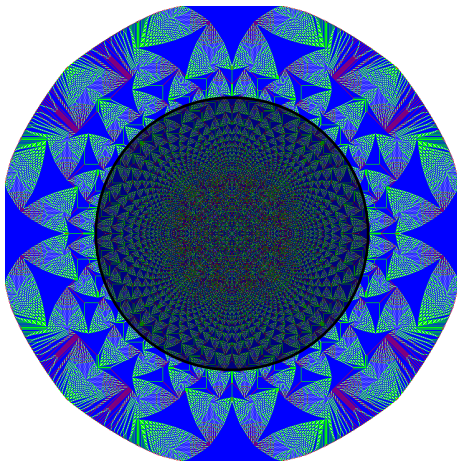
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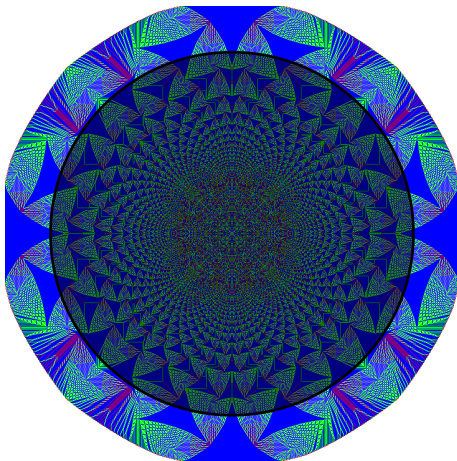
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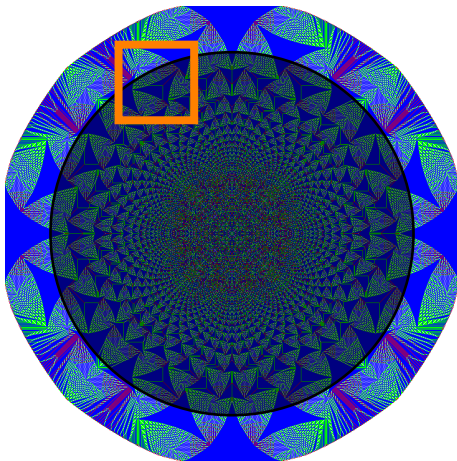
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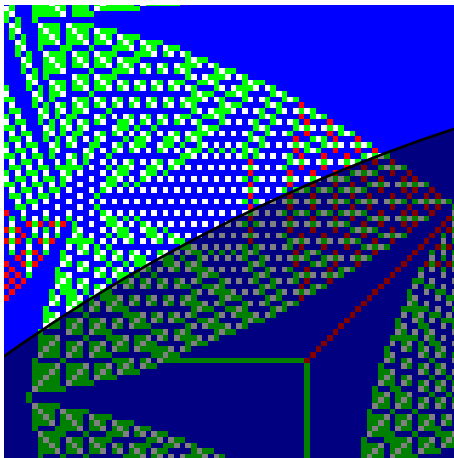
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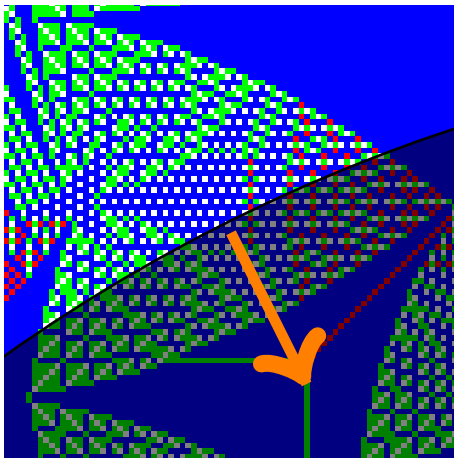
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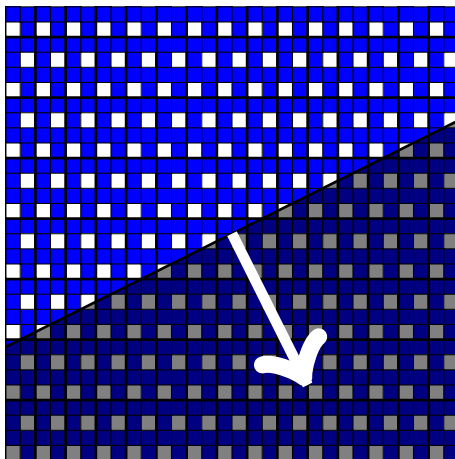
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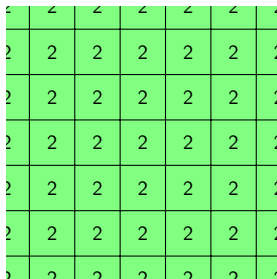
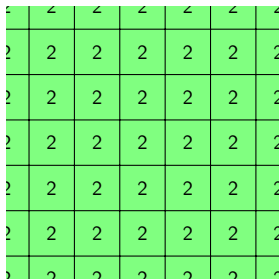


Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?

Heuristic: locally, toppling the sink behave as the toppling of an half-plane

Definition (Weak Dhar Criterion [D., Le Borgne 2018])

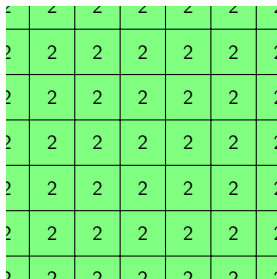
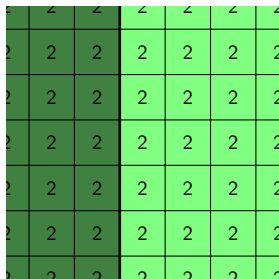
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).



← Direction $\vec{s}$ du puits

Definition (Weak Dhar Criterion [D., Le Borgne 2018])

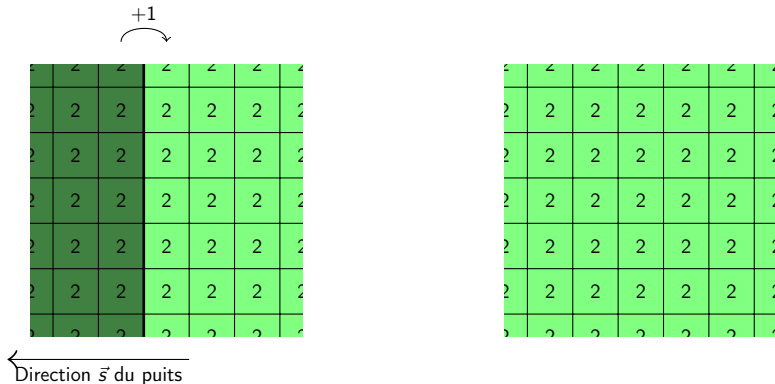
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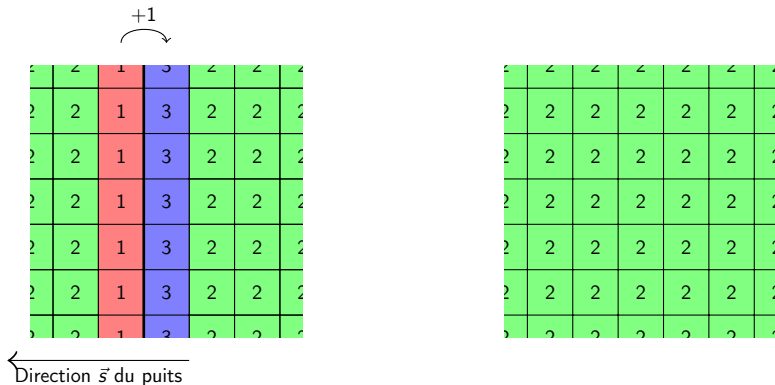
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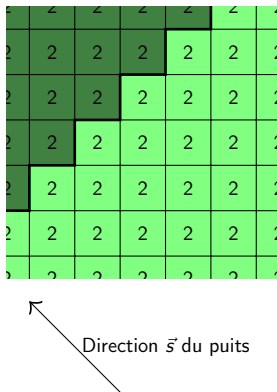
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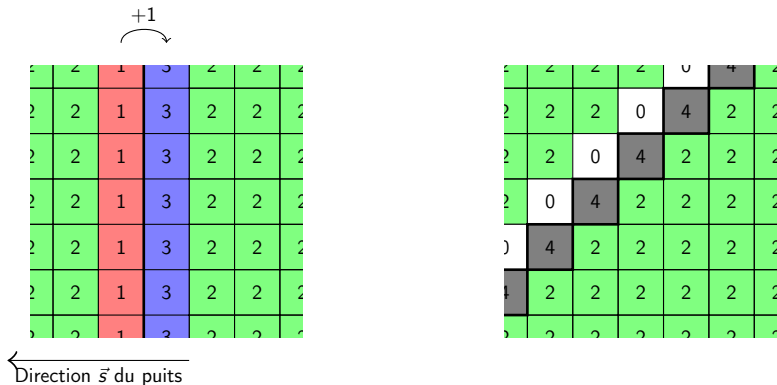
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

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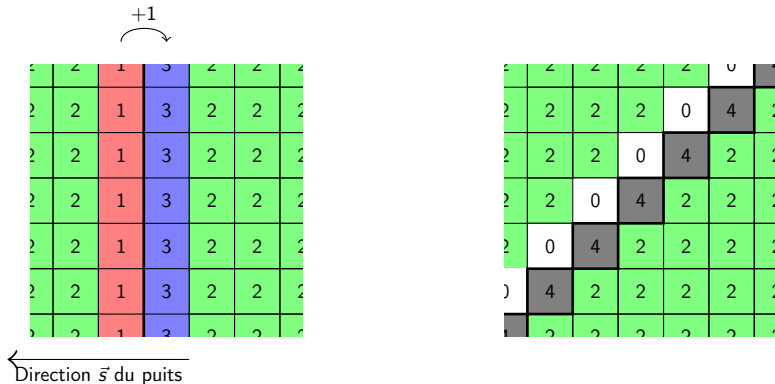
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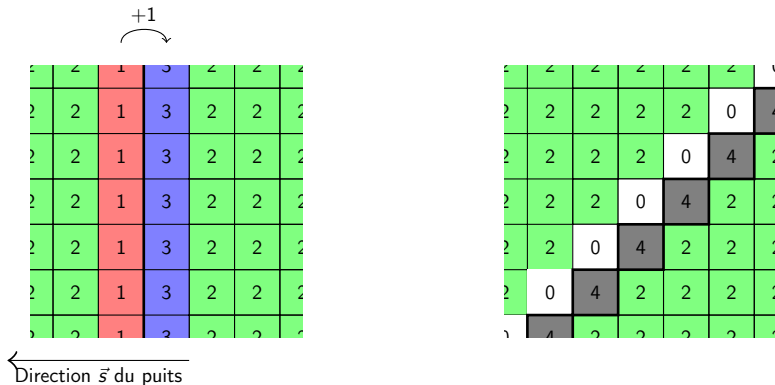
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).



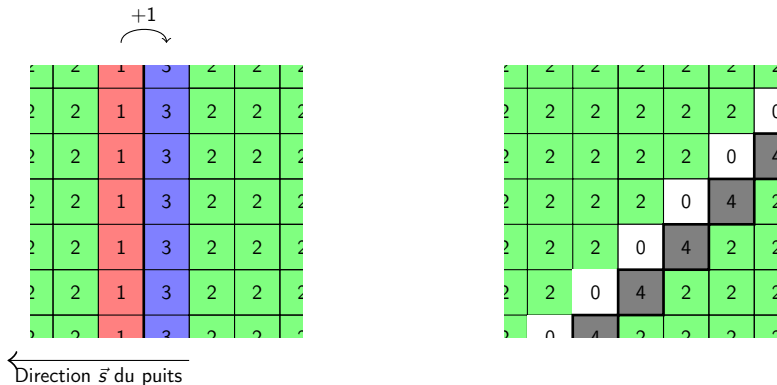
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).



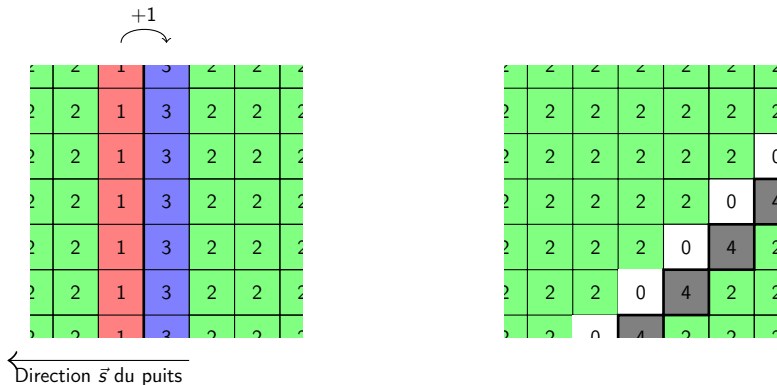
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).



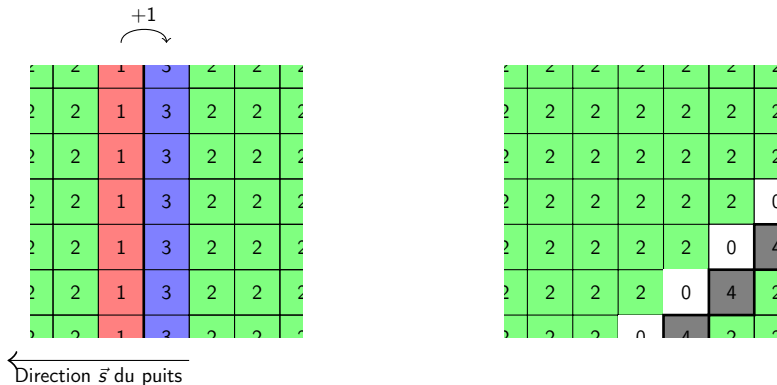
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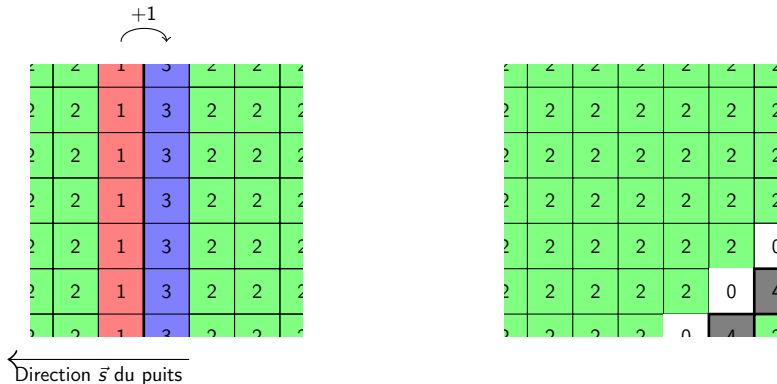
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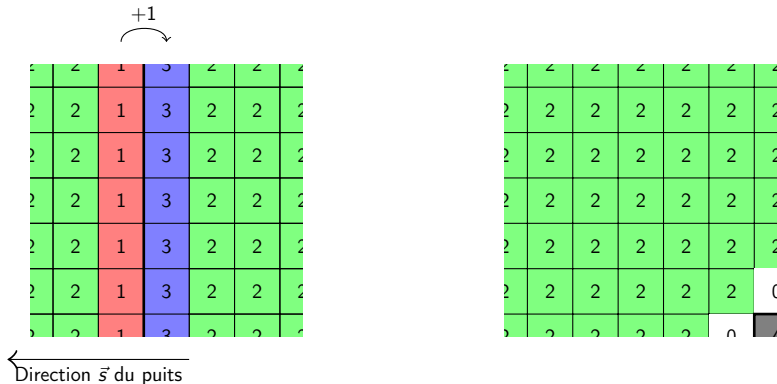
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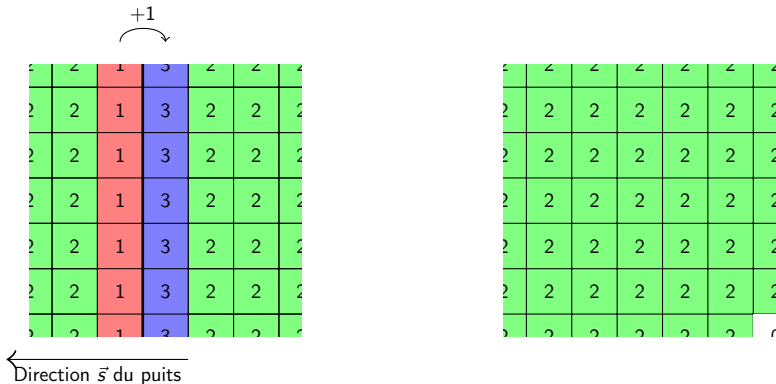
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

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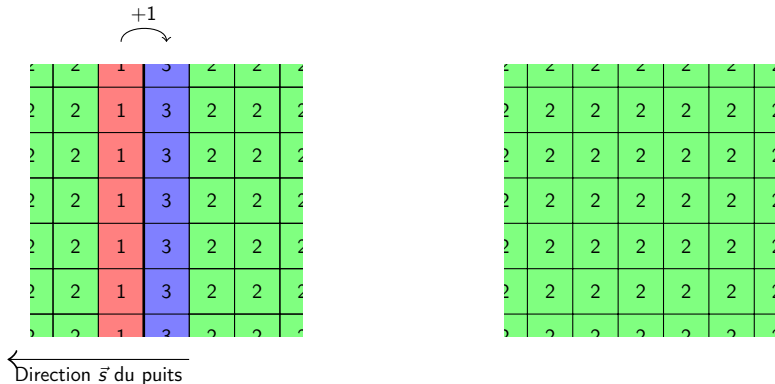
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).



Definition (Weak Dhar Criterion [D., Le Borgne 2018])

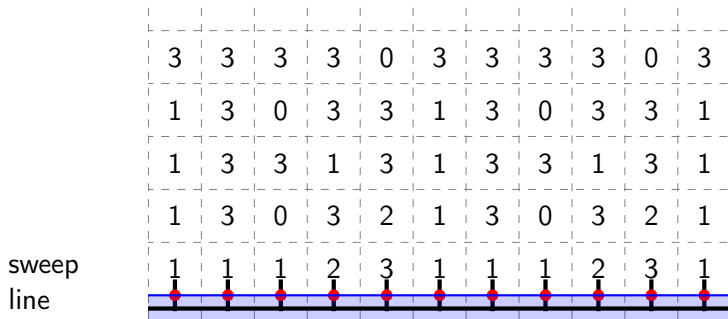
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).



Demo

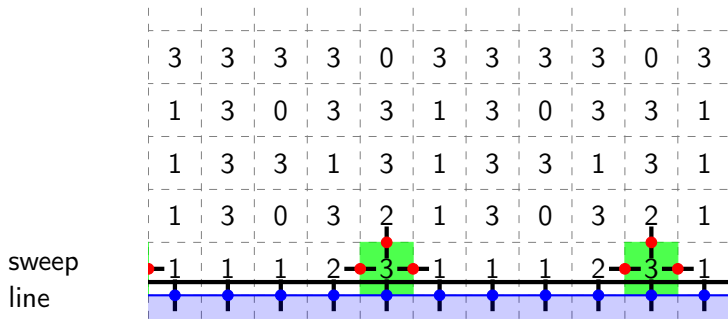
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



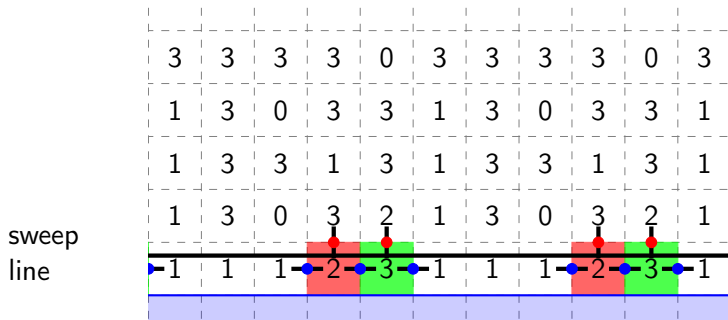
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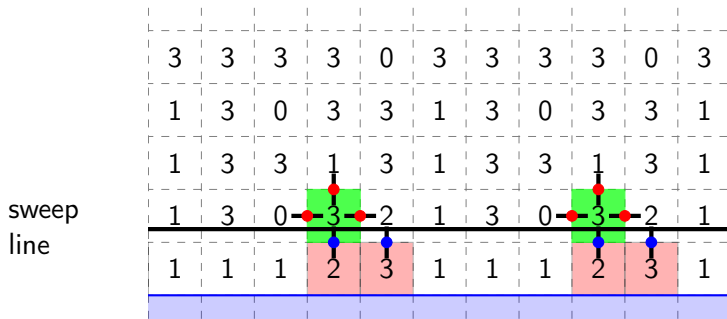
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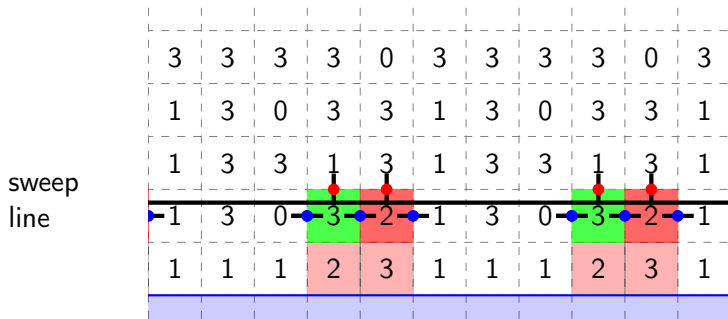
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



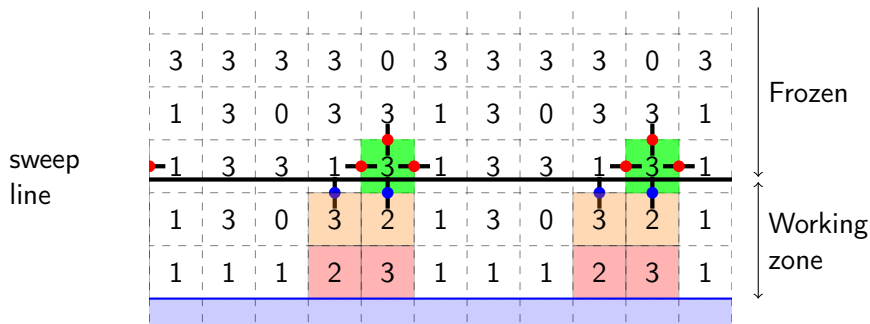
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



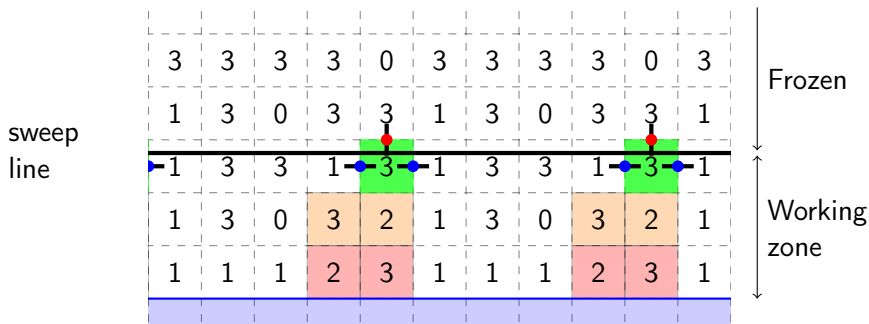
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



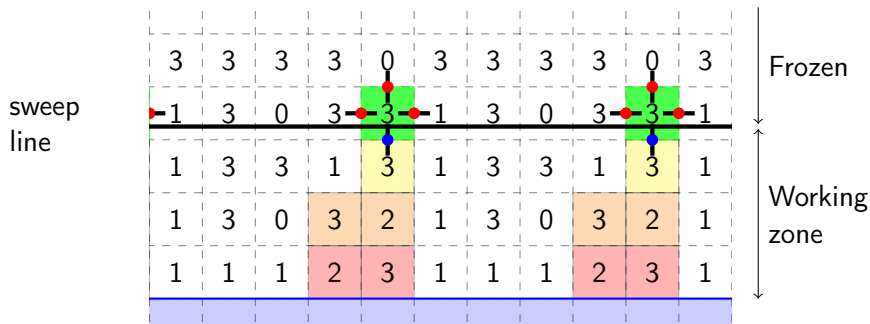
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



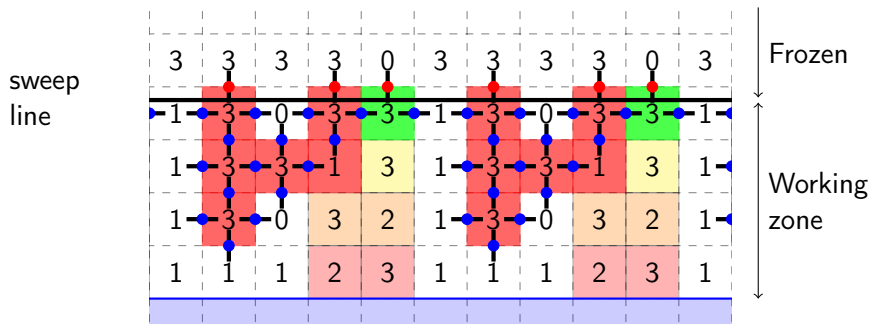
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



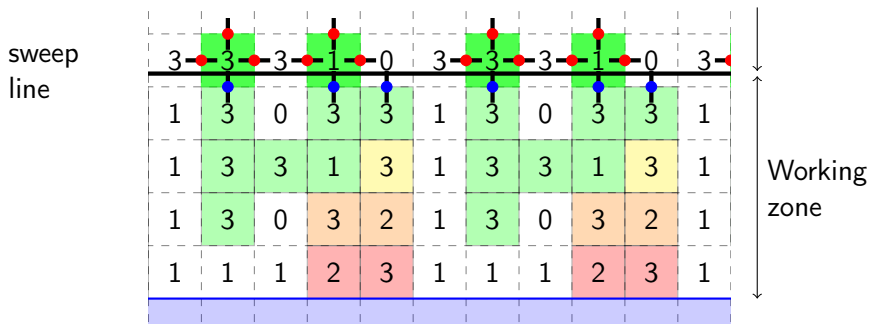
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



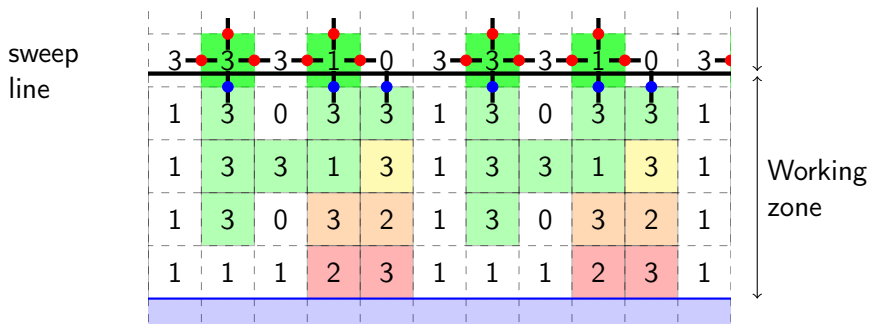
Theorem (D., Le Borgne 2018)

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Theorem (D., Le Borgne 2018)

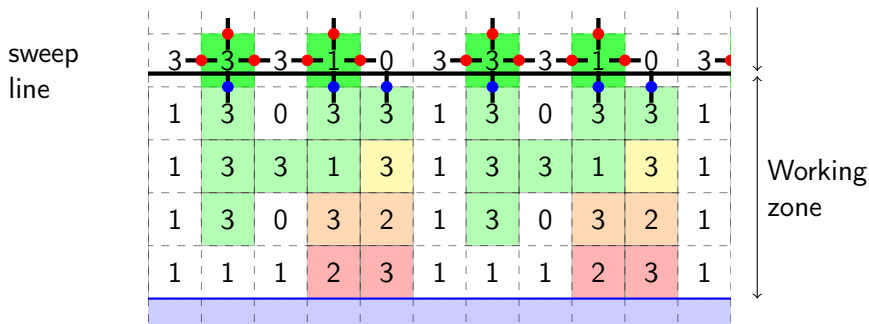
The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



- Periodicity along the orthogonal of the sink

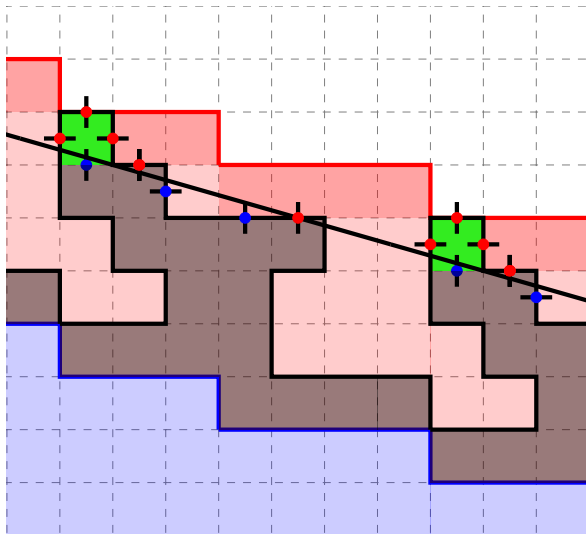
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



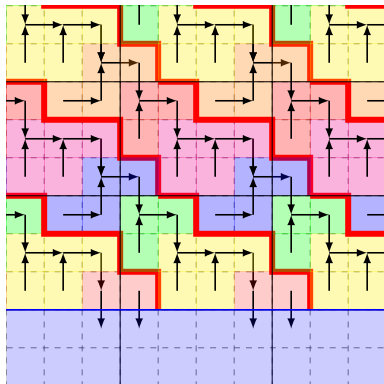
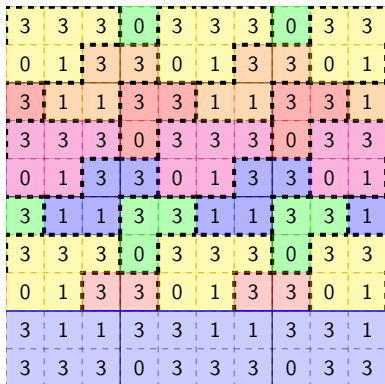
- ▶ Periodicity along the orthogonal of the sink
- ▶ Ultimately periodicity in the opposite direction of the sink, whatever the starting half-plane

sweep line

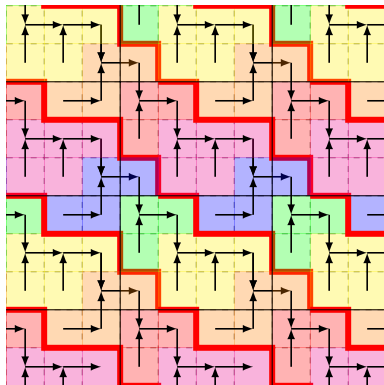
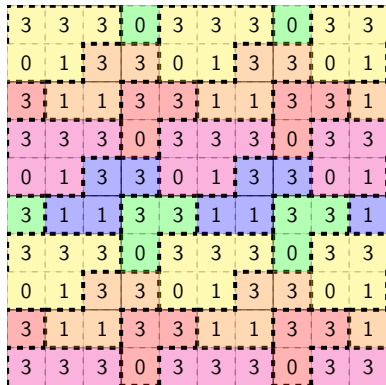


3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3

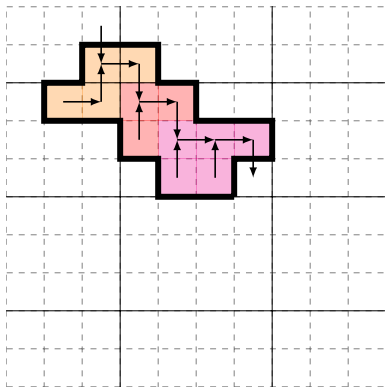
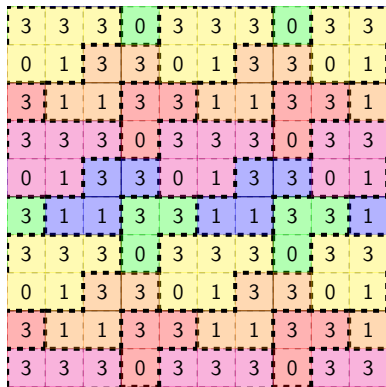
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3



Periodic spanning forest rooted on the half-plane

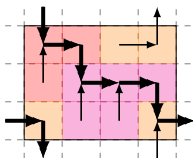


Biperiodic spanning forest with infinite paths directed towards the sink



Biperiodic spanning forest with infinite paths directed towards the sink

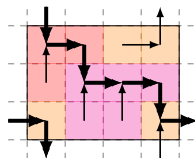
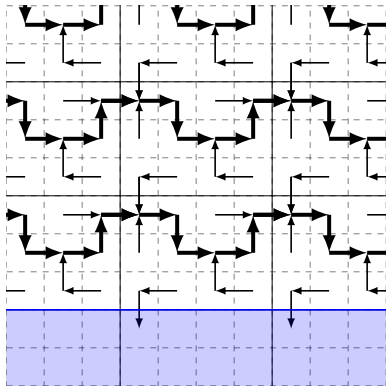
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3



Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$

Theorem [D., Le Borgne 2018]

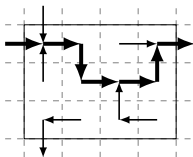
Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $\mathcal{F}_{W \times H, \vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.



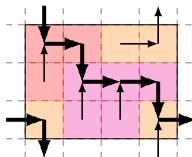
Spanning forests of the torus rooted
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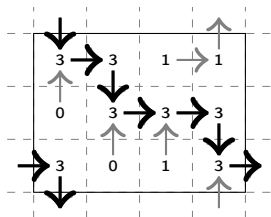
Spanning forest of the torus with slope $(1, 0)$ incompatible with the vertical direction



Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$

Theorem [D., Le Borgne 2018]

Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $\mathcal{F}_{W \times H, \vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.



Determinantal formula [Kenyon 17] for non contractible cycle rooted spanning forests (NCRSFs)

Refinement with the infinite path's slope

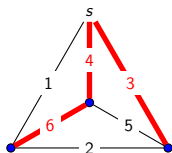
$k \cdot j$	$k \cdot i$				
	0	1	2	3	4
0		31300528	541732	1528	1
1	31300528	5427200	31232	4	
2	541732	31232	6		
3	1528	4			
4	1				

Table: Number of NCRSFs with k cycles of slope (i, j) on the torus $T_{4,4}$

Computation for $W, H \leq 9$

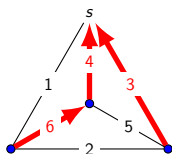
Inverse function

Placing the grains on the edges. ● ○



Inverse function

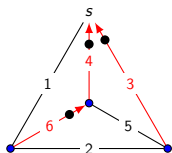
Placing the grains on the edges. ● ○



► Orientation towards the sink

Inverse function

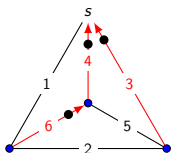
Placing the grains on the edges. ● ○



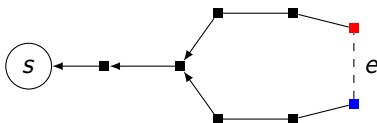
- ▶ Orientation towards the sink
- ▶ Internal: 1 grain ● to the father

Inverse function

Placing the grains on the edges. ● ○

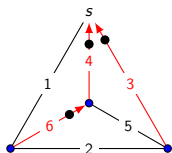


- ▶ Orientation towards the sink
- ▶ Internal: 1 grain ● to the father
- ▶ External: ● depends on the position of the maximal edge on the fundamental cycle

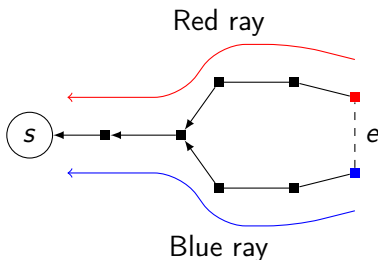


Inverse function

Placing the grains on the edges. ● ○

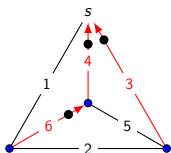


- ▶ Orientation towards the sink
- ▶ Internal: 1 grain ● to the father
- ▶ External: ● depends on the position of the maximal edge on the fundamental cycle

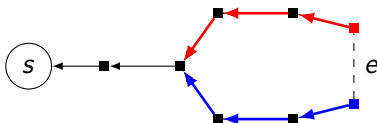


Inverse function

Placing the grains on the edges. ● ○

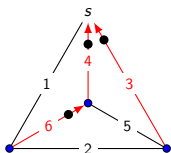


- ▶ Orientation towards the sink
- ▶ Internal: 1 grain ● to the father
- ▶ External: ● depends on the position of the maximal edge on the fundamental cycle

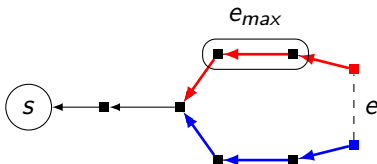


Inverse function

Placing the grains on the edges. ● ○

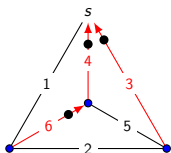


- ▶ Orientation towards the sink
- ▶ Internal: 1 grain ● to the father
- ▶ External: ● depends on the position of the maximal edge on the fundamental cycle

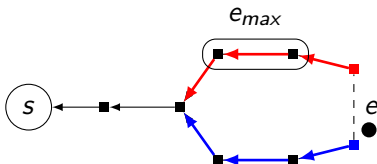


Inverse function

Placing the grains on the edges. ● ○

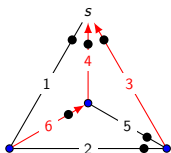


- ▶ Orientation towards the sink
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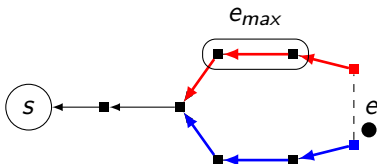


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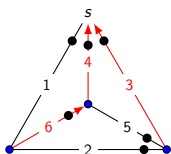


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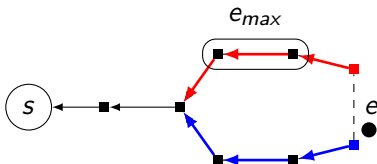


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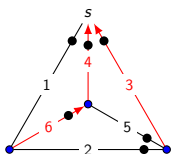


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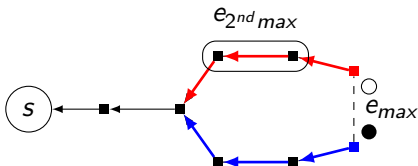


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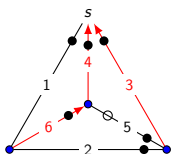


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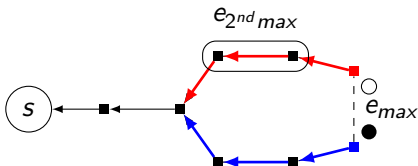


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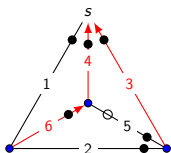


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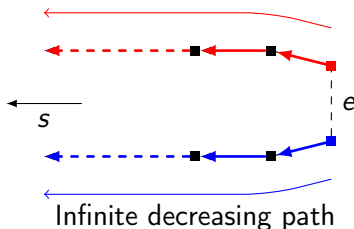
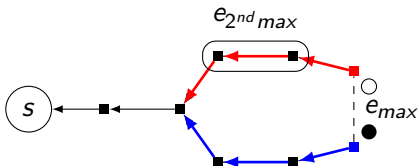


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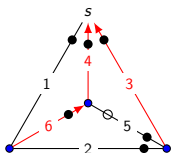
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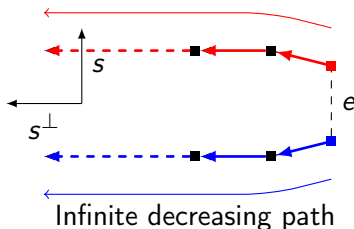
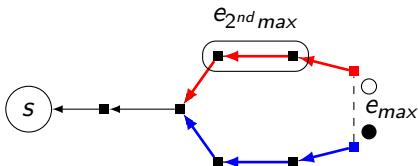
Cycles are directed such that they are globally decreasing.
Periodicity $\Rightarrow$ Maximal edge at finite distance

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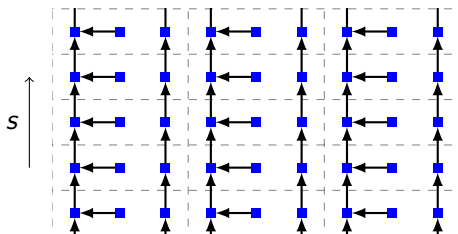
Restricted Tutte Polynomial

$$\mathcal{T}_{W \times H, s}(x, y) = \sum_{T \in \mathcal{F}_{W \times H}} x^{\text{int}_{W \times H}(T)} y^{\text{ext}_{W \times H}(T)}$$

$e <_s f$ if e is closer to the sink than f .

Restrictions

- ▶ On NCRSF: $\mathcal{F}_{W \times H}$.
- ▶ On the activity: on the rectangular fundamental domain $W \times H$ consider exactly $2WH$ edges.



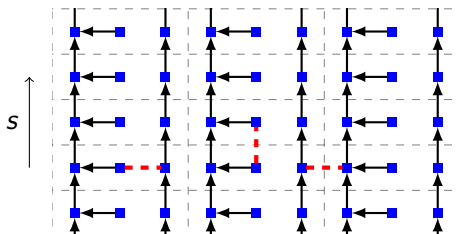
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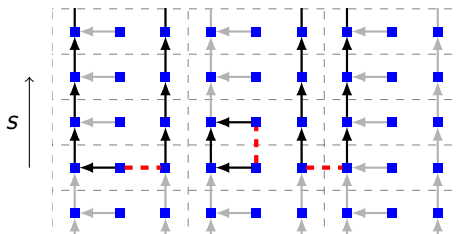
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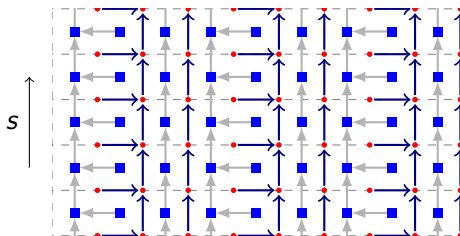
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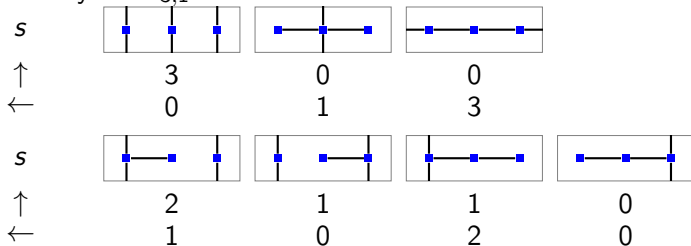
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Restricted Tutte Polynomial

External activity on $\mathcal{F}_{3,1}$:

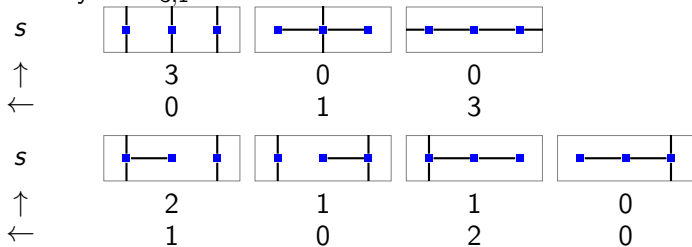


$$\mathcal{T}_{3 \times 1, (0,1)}(1, y) = y^3 + 3y^2 + 6y + 7$$

$$\mathcal{T}_{3 \times 1, (-1,0)}(1, y) = y^3 + 3y^2 + 6y + 7$$

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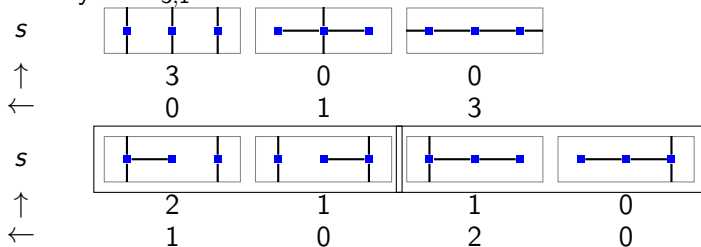
$$\mathcal{T}_{3 \times 1, (-1,0)}(1, y) = y^3 + 3y^2 + 6y + 7$$

Theorem (D., Le Borgne 2018)

For any directions s, s' , $\mathcal{T}_{W \times H, s}(1, y) = \mathcal{T}_{W \times H, s'}(1, y)$.

Restricted Tutte Polynomial

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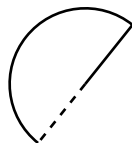
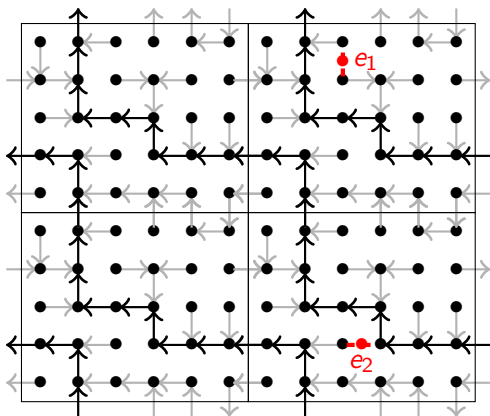
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Since $\mathbb{Z}^2$ is self-dual, we have:

$$\mathcal{T}_{3 \times 1, (0,1)}(x, y) = x^3 y^3 + 3xy^2 + 3x^2 y + 3x + 3y + 4$$

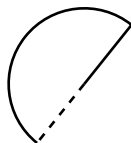
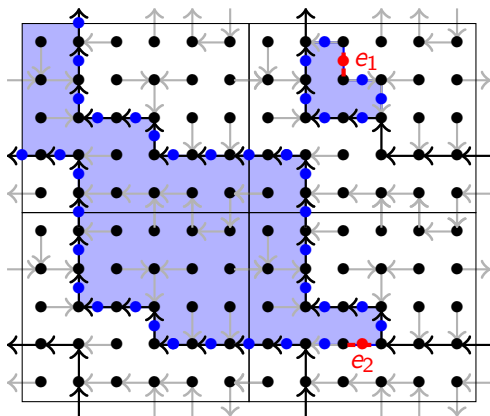
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External activity



Direction of the sink

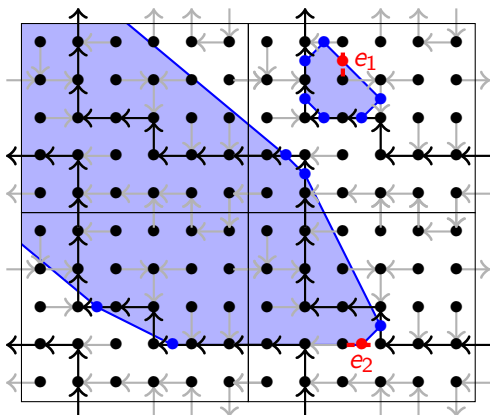
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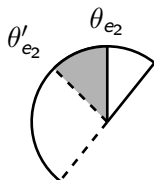
Direction of the sink

- Convex hulls of fundamental cycles.

External activity



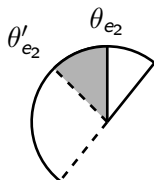
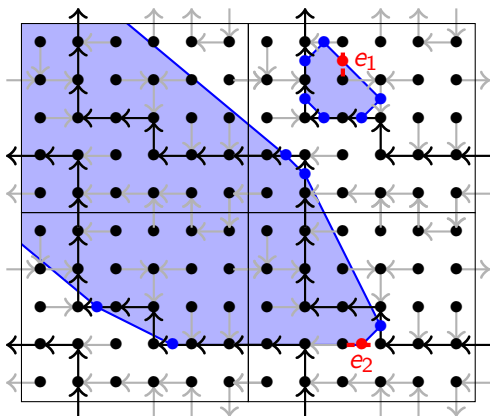
For each external edge e , there is an activity sector $[\theta_e, \theta'_e)$.



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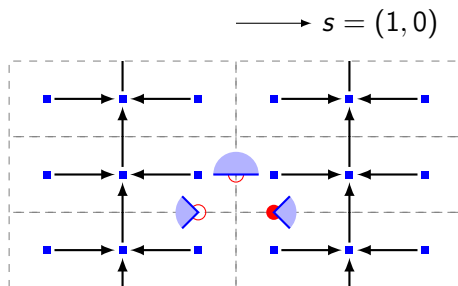
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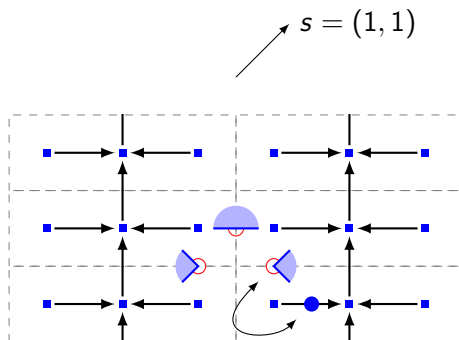
For each external edge e , there is an activity sector $[\theta_e, \theta'_e)$.

For any sector excluding all $(\theta_e)_e$ and $(\theta'_e)_e$, the external activity is invariant.

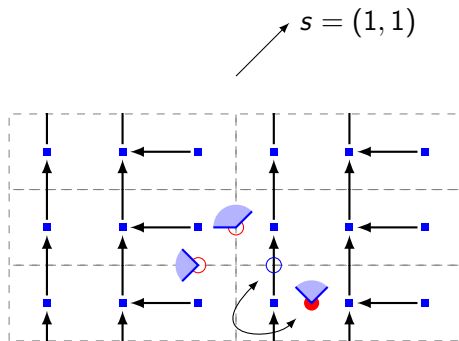
Critical pair exchange : Rotation step



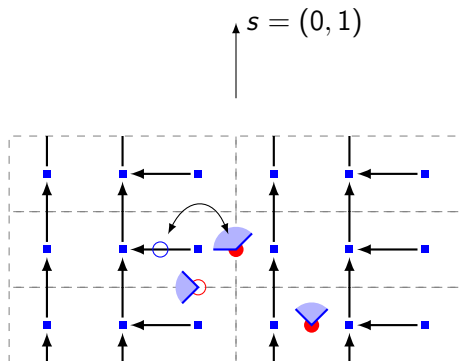
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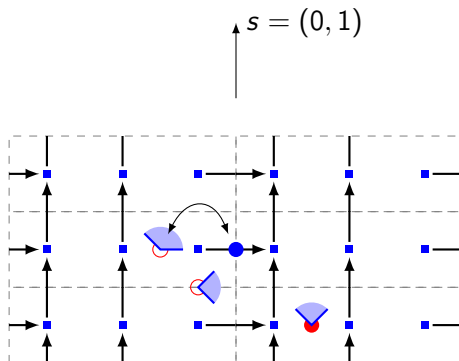
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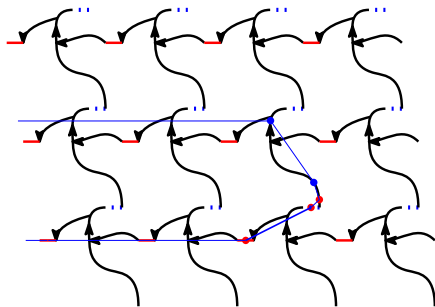
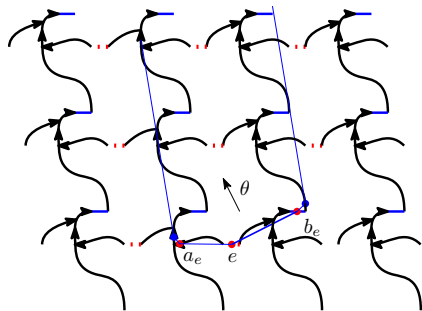
Critical pair exchange : Rotation step



Critical pair exchange : Rotation step



Critical pair exchange: changing forest slope



Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	▷ Weak Dhar Criterion (projective sink)
▷ Bijection between recurrent and spanning trees	▷ Bijection <i>recurrent</i> and some spanning forests of the torus
▷ Tutte polynomial	▷ Restriction of Tutte polynomial
▷ Invariant by edge exchange	▷ Distribution of external activity invariant by rotation of projective sink
▷ Symmetric for self-dual planar graphs	▷ Symmetric joint distribution of external/internal activities <i>changing</i> by rotation

Conclusion

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- ▶ Involution on NCRSFs for atomic rotation preserving this distribution

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Perspectives

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 $e <_E f \Rightarrow e + (iW, jH) <_E f + (iW, jH)$ and
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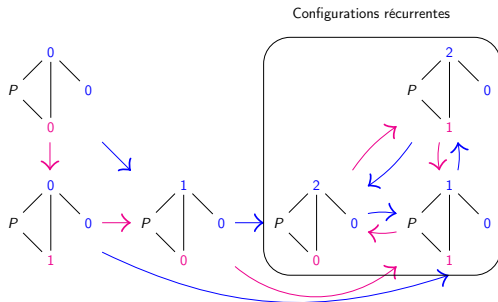
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 - ▶ Only decreasing, or only periodic
 - ▶ Anything else

THANK YOU

Markov Chain for $G = (V \cup \{S\}, E)$

- ▶ States: stable configurations on G
- ▶ Transition: Add a particle to a vertex chosen uniformly and stabilize



- ▶ The recurrent states are called recurrent configurations.
- ▶ The stationary distribution is uniform on the recurrent configurations.

Dhar Criterion A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)

Close to [Pegden and Smart, 2017]

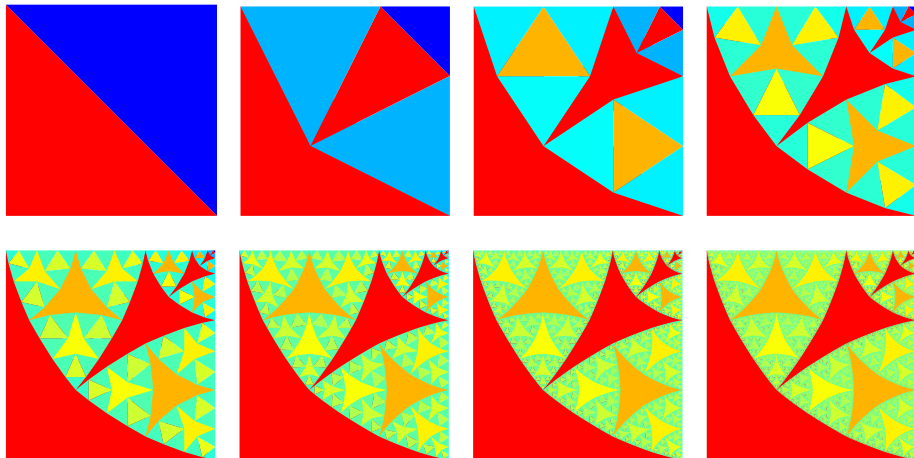


Figure: Each non blue zone is described by a quadratic form. [arxiv:1708.09432]

Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

$$M(a, b, c) = \begin{pmatrix} c + a & b \\ b & c - a \end{pmatrix}$$

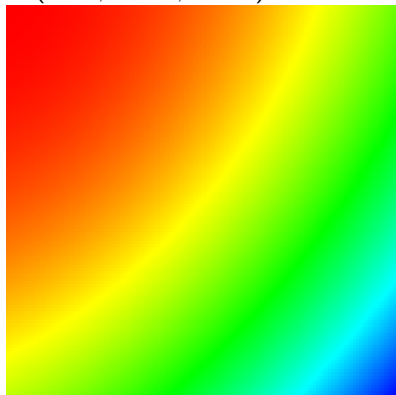
The number of topples is:

$$\begin{aligned} h(\mathbf{x}) &= \left\lceil \frac{1}{2} \mathbf{x}^t M(a, b, c) \mathbf{x} \right\rceil \\ &= (c + a)x^2 + 2bxy + (c - a)y^2 \end{aligned}$$

Sample

$M(0.25, 0.875, 2.125)$

for



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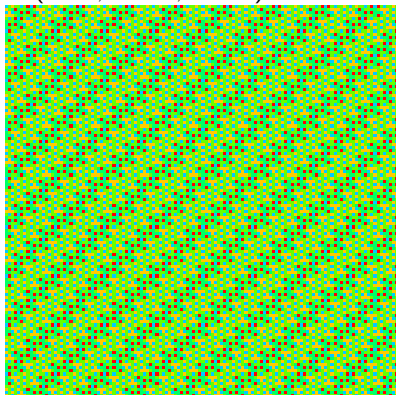
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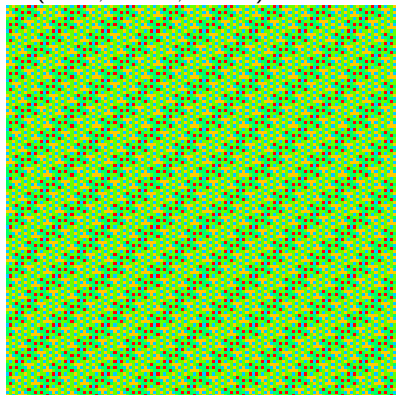
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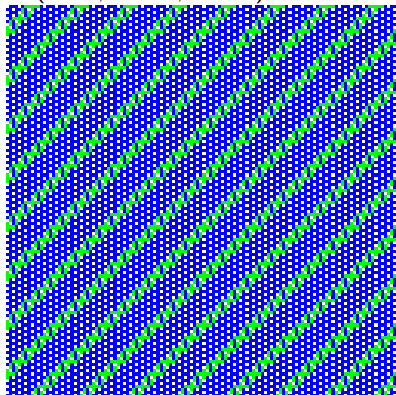
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- ▶ It's *periodic* for $a, b, c \in \mathbb{Q}$
- ▶ But it may be negative and/or unstable!

Sample

for

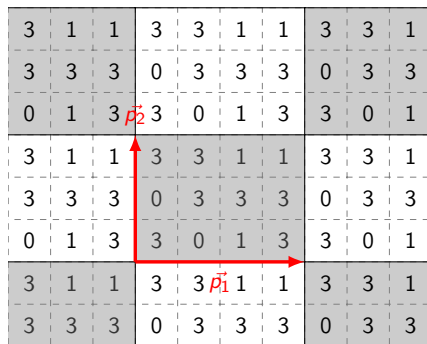
$M(0.25, 0.875, 2.125)$



A definition of recurrence for periodic stable configurations

Pattern + two dimensional period $(\vec{p}_1, \vec{p}_2)$.

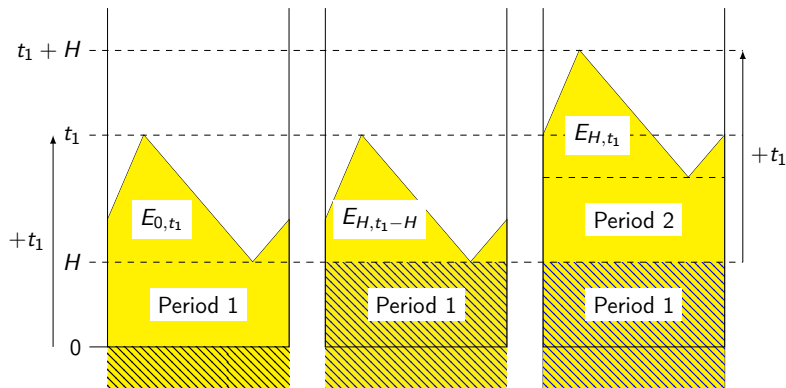
$$\forall \mathbf{x} \in \mathbb{Z}^2 u(\mathbf{x}) = u(\mathbf{x} + \vec{p}_1) = u(\mathbf{x} + \vec{p}_2)$$



Periodicity on $-s$

Lemme

If a periodic configuration is recurrent, then there exists a position $y = t_1$ for which all vertices of the first period are toppled.



We have $Period1 \subset E_{0,t_1} \Rightarrow E_{0,t_1} = E_{H,t_1-H}$ and $v \in E_{0,t_1} \Rightarrow v + H\vec{y} \in E_{H,t_1}$. Then $E_{H,t_1} \supset Period2$.