

# Staircases and dominoes: Bounding the growth rate of 1324-avoiding permutations.

Andrew Elvey Price

Joint work with David Bevan, Robert Brignall and Jay Pantone.

The University of Melbourne (and LaBRI)

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# PATTERN AVOIDING PERMUTATIONS

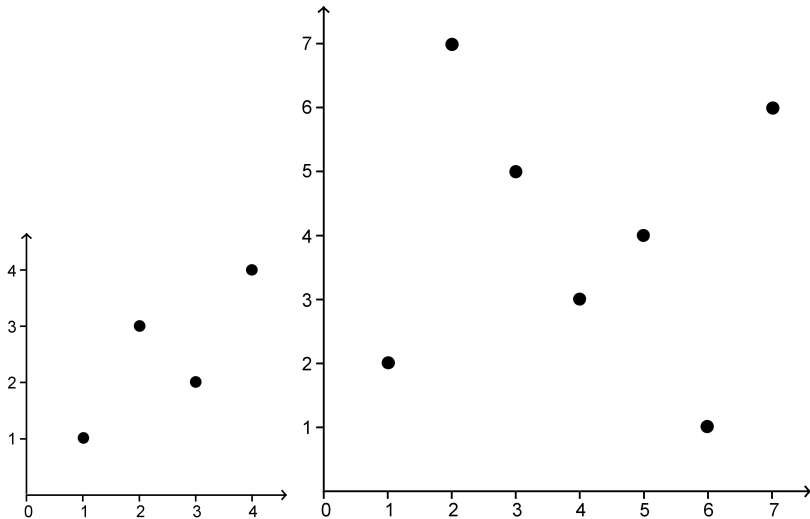
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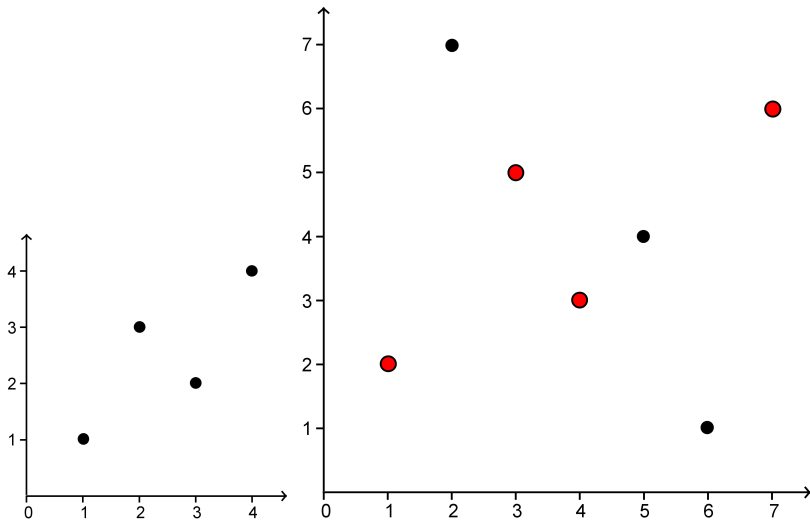
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- We are mostly interested in the behaviour of the sequence  $|\text{Av}_n(\pi)|$  for each permutation  $\pi$ .

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- Remarkably, these are all Wilf equivalent as

$$|\text{Av}_n(123)| = |\text{Av}_n(132)| = \frac{1}{n+1} \binom{2n}{n},$$

the  $n$ th Catalan number.

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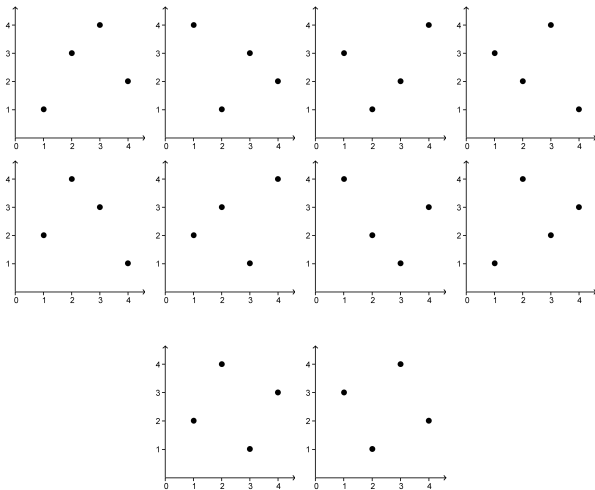
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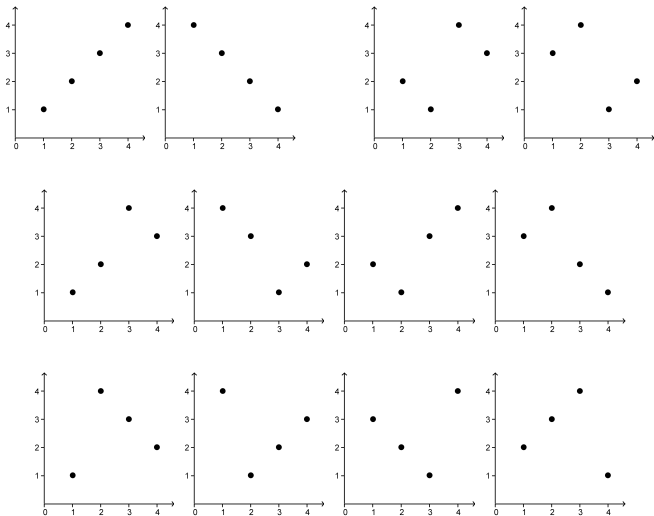
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- The sequence  $|Av_n(1324)|$  is a mystery.

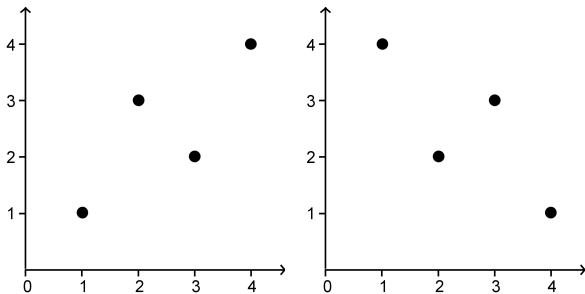
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This is called the Stanley-Wilf limit of  $\pi$ .

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- Very few other Stanley-Wilf limits are known.



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- Fox (2013):  
For  $\alpha < 1/4$ , almost all permutations  $\pi$  satisfy

$$\text{gr}(\text{Av}(\pi)) > 2^{|\pi|^\alpha}.$$

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- Tony Guttmann’s analysis of this series suggests that

$$|Av_n(1324)| \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^\theta,$$

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- The presence of the  $\mu_1^{n^\sigma}$  term suggests that the sequence  $Av_n(1324)$  is not D-finite.

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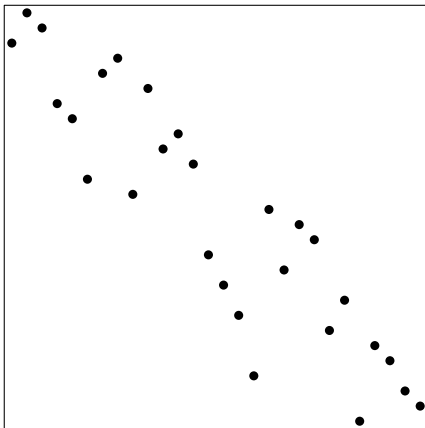
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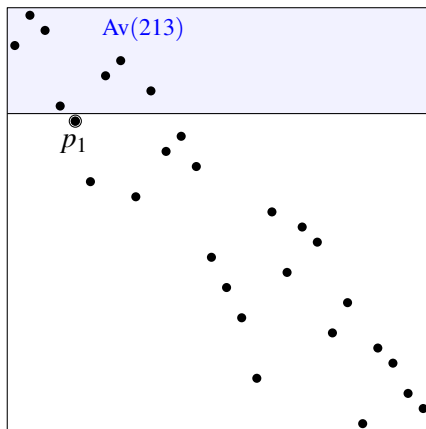
Theorem (Bevan, Brignall, E., Pantone)

$$10.27 \leq \text{gr}(\text{Av}(1324)) \leq 13.5.$$

# STAIRCASING A 1324-AVOIDER

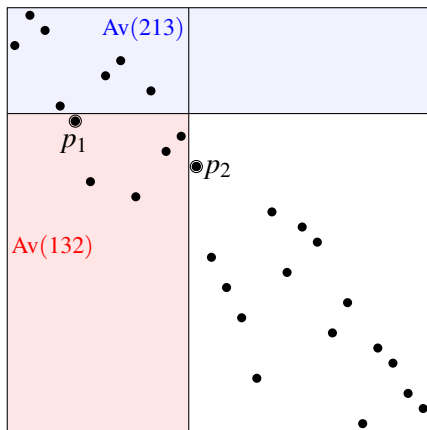


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- $p_1$  uppermost 1 in a 213.

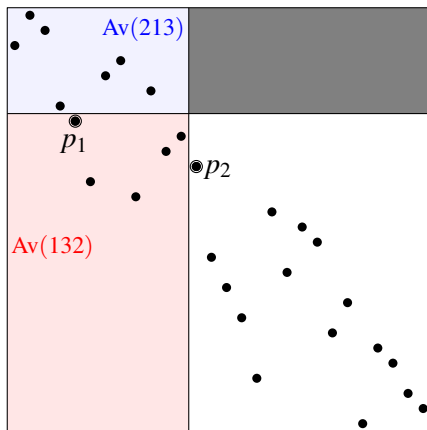
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- $p_1$  uppermost 1 in a 213.
- $p_2$  leftmost 2 in a 132 consisting of points below  $p_1$  divider.

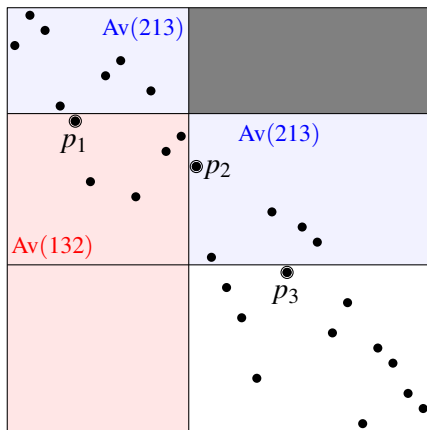


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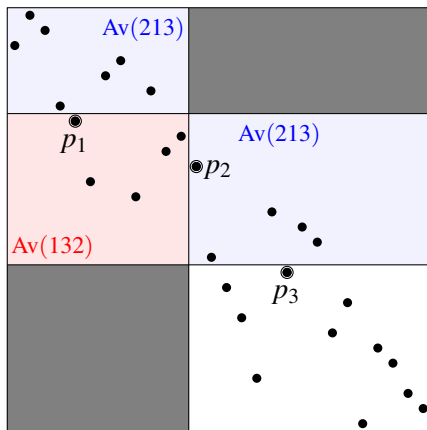
- $p_2$  leftmost 2 in a 132 consisting of points below  $p_1$  divider.
- No points above  $p_1$  and to the right of  $p_2$ .

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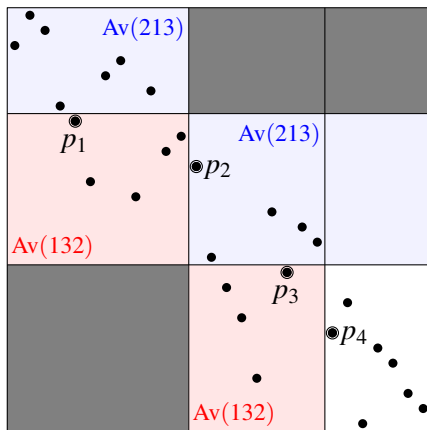
- $p_2$  leftmost 2 in a 132 consisting of points below  $p_1$  divider.
- $p_3$  uppermost 1 in a 213 consisting of points to right of  $p_2$  divider.

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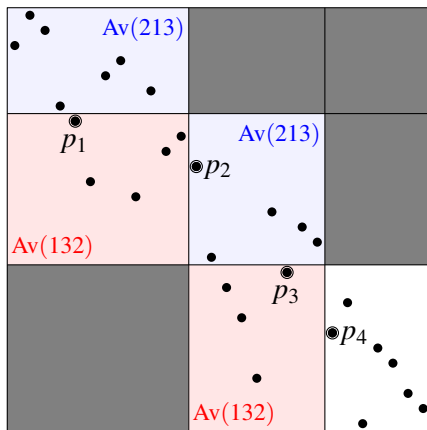
- $p_3$  uppermost 1 in a 213 consisting of points to right of  $p_2$  divider.
- No points to the left of  $p_2$  and below  $p_3$ .

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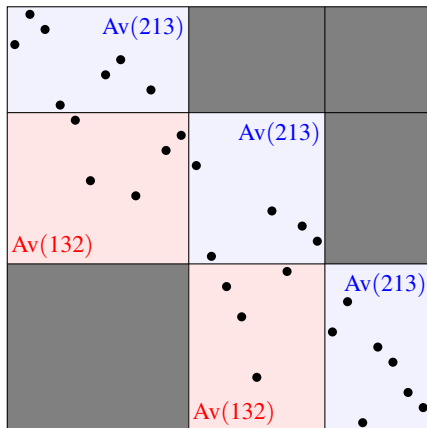
- $p_3$  uppermost 1 in a 213 consisting of points to right of  $p_2$  divider.
- $p_4$  leftmost 2 in a 132 consisting of points below  $p_3$  divider.

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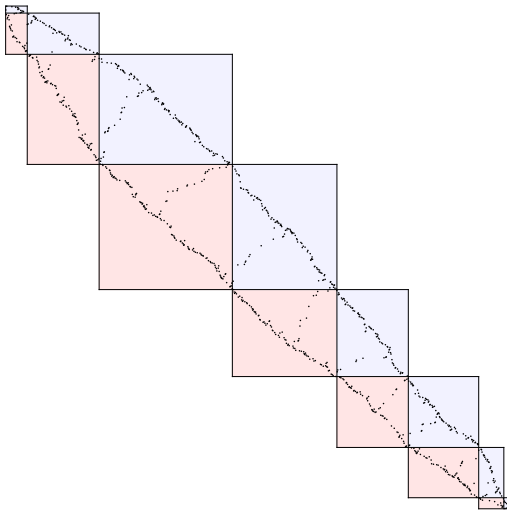
- $p_4$  leftmost 2 in a 132 consisting of points below  $p_3$  divider.
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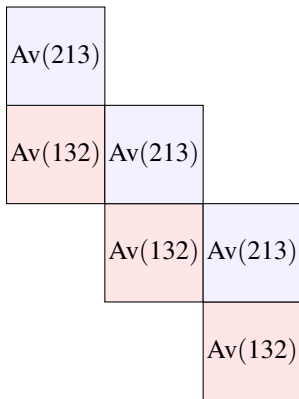
- The final cell avoids 213.

# A LARGE STAIRCASED 1324-AVOIDER



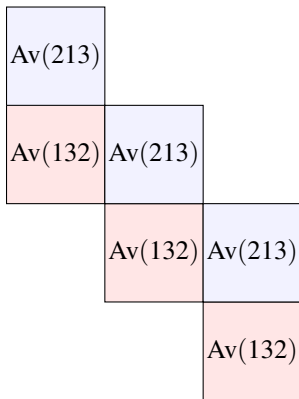
Data provided by Einar Steingrímsson.

# UPPER BOUND ON $\text{GR}(\text{Av}(1324))$





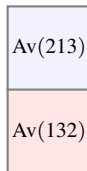
# UPPER BOUND ON $\text{GR}(\text{Av}(1324))$



This already gives us an upper bound:  $|\text{Av}_n(1324)|$  is less than the number of staircases with  $n$  points, which grows like  $16^n$ .

# DOMINOES

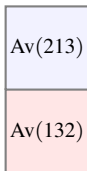
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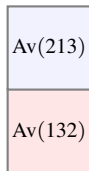


that avoids 1324.

That is, a 1324-avoiding staircase with two cells.

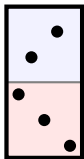
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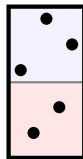
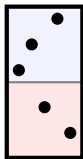


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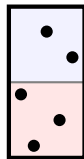
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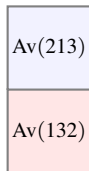


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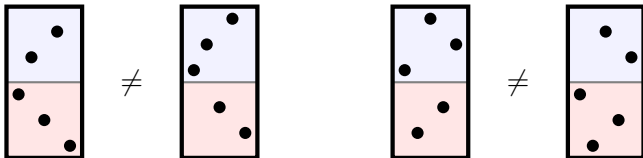
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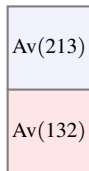
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Claim: The number of  $n$ -point dominoes is:

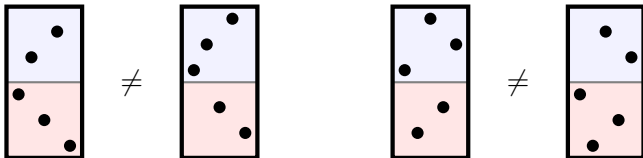
# DOMINOES

A domino is a *gridded permutation* in



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Claim: The number of  $n$ -point dominoes is: 
$$\frac{2(3n + 3)!}{(n + 2)!(2n + 3)!}.$$

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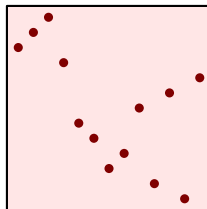
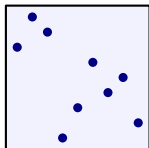
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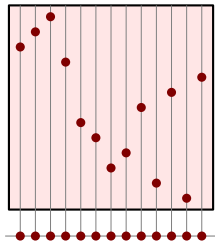
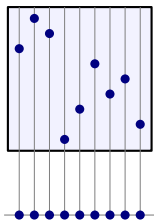
# BIJECTING 213/132-AVOIDERS TO ARCH SYSTEMS



A 213-avoider and a 132-avoider

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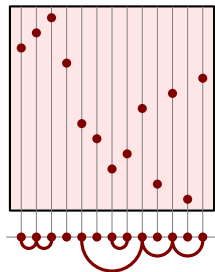
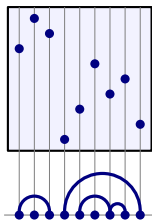


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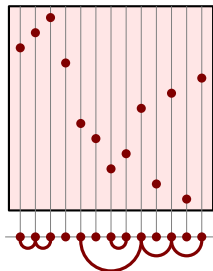
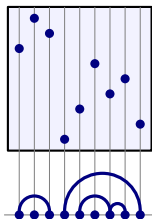
- vertically project the points in the permutation.
- draw an arch between two points when the right point in the permutation is one higher than the left point.



A 213-avoider and a 132-avoider with their arch systems

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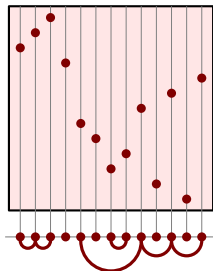
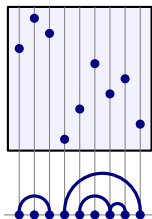
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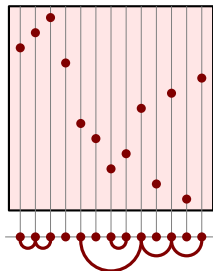
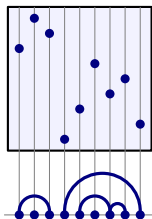
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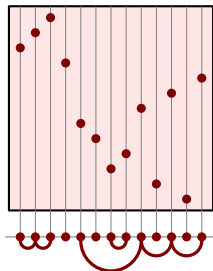
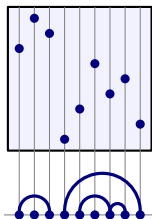
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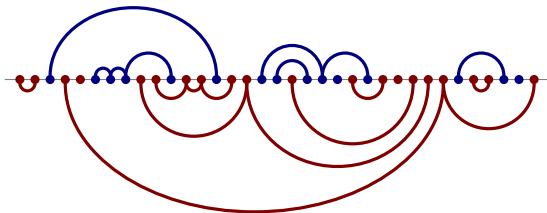
- Each point is at the left end of at most one arch.
- Each point is at the right end of at most one arch.
- No two arches cross.




# BIJECTING DOMINOES TO ARCH CONFIGURATIONS.

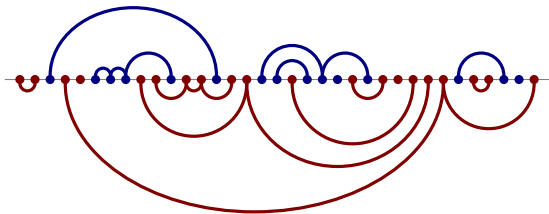
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- The 1324-avoiding condition corresponds to forbidding the pattern  in the arch configuration.





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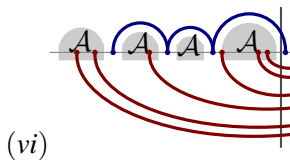
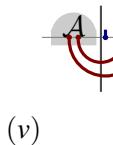
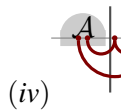
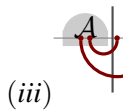
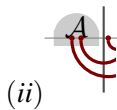
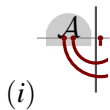
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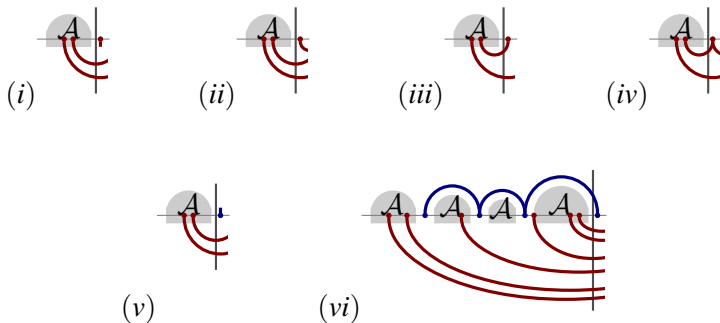
Then the number of dominoes with  $n$  points is  $[t^n]A(t, 0)$ .



# DECOMPOSITION OF ARCH SYSTEM PREFIXES



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This decomposition leads to the equation:

$$A(z) = tA(z) + tzA(z) + t(1+z)\frac{A(z) - A(0)}{z} + \frac{1}{1-tA(z)}.$$

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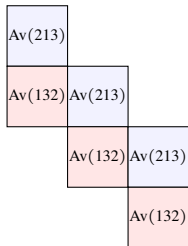
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- Now we have enumerated dominos!
- We can use these dominoes to get better upper and lower bounds for  $\text{gr}(\text{Av}(1324))$

# UPPER BOUND ON $\text{GR}(\text{AV}(1324))$



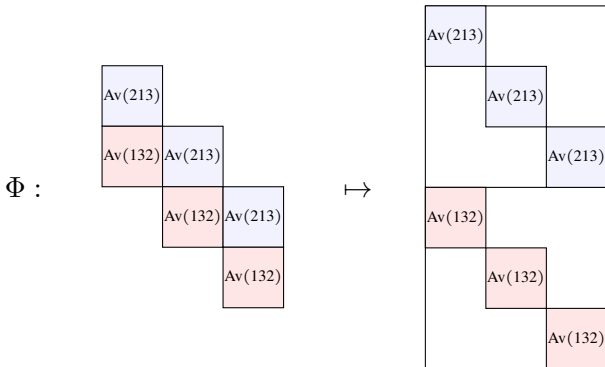
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- Start with a 1324-avoiding permutation, and staircase it.



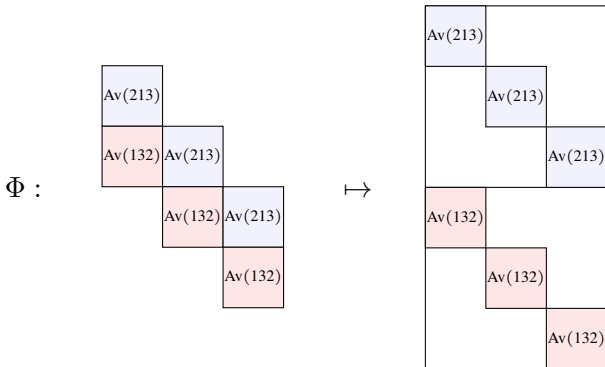
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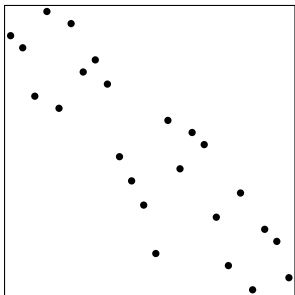
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- $\Phi$  is not injective.



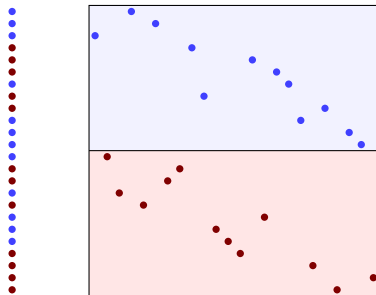


# UPPER BOUND ON $\text{GR}(\text{AV}(1324))$

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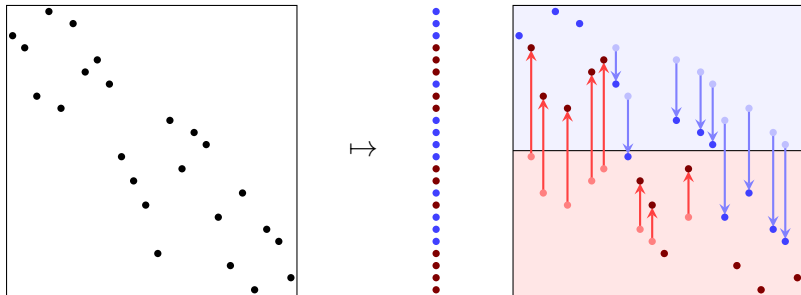


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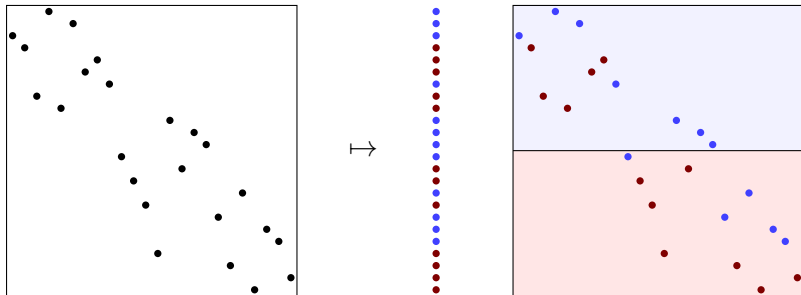
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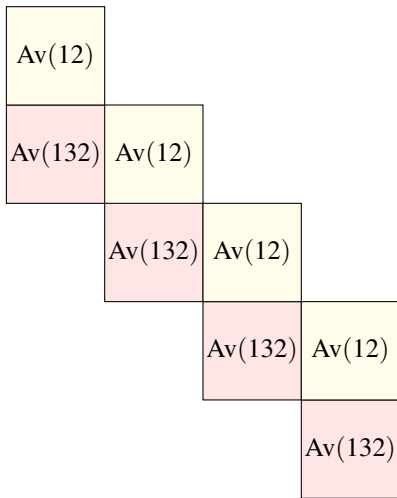


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This staircase always defines a 1324-avoider.

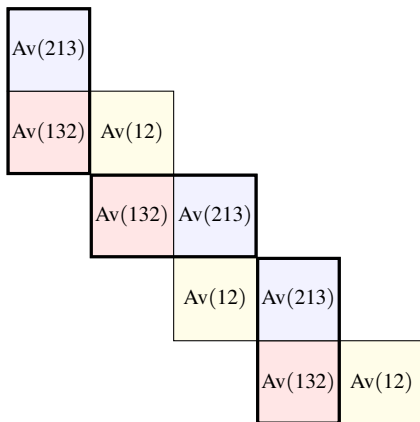
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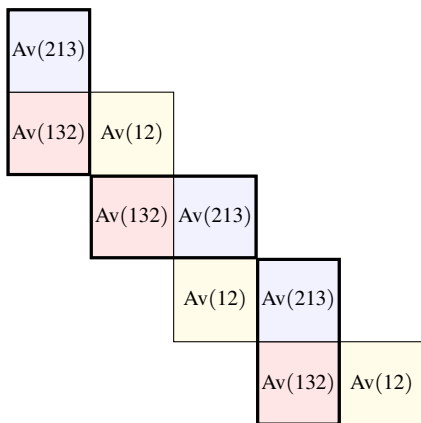
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- We can enumerate these exactly: their exponential growth rate is 9.

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- Count 1324-avoiding staircases in which every third cell is 12-avoiding.

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- Taking limits carefully, this gives a lower bound for  $\text{gr}(\text{Av}(1324))$  of 10.125.

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- There is still a long way to go to get near Guttman's estimate of 11.6.

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THANK YOU