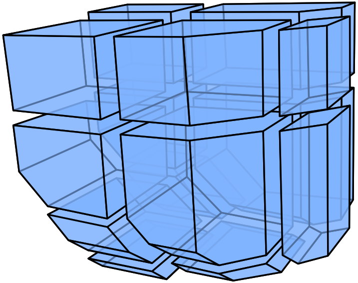


ν-Associahedra

Arnau Padrol
IMJ - PRG
Sorbonne Université

reporting joint work with

Cesar Ceballos and
Camilo Sarmiento

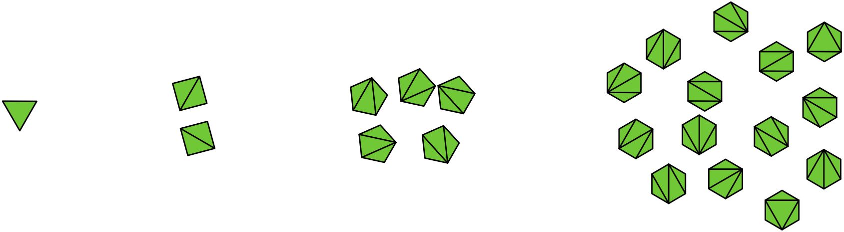
Journées de Combinatoire de Bordeaux - 12/02/2018

The ν-Tamari lattice

Catalan numbers

The *Catalan numbers*

$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$



$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

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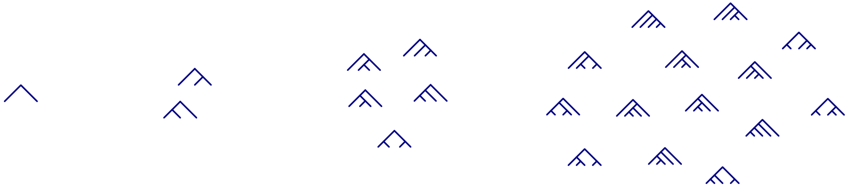

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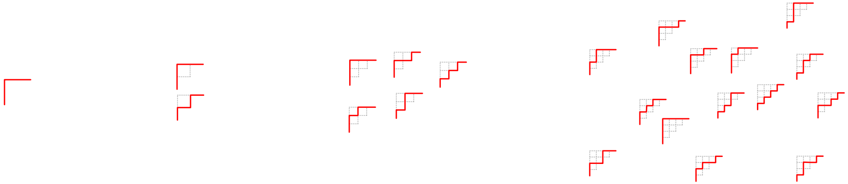
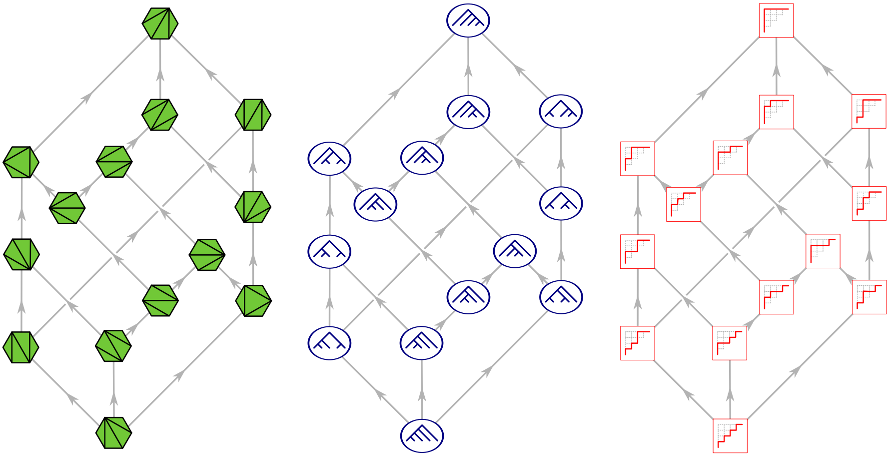


$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

The Tamari lattice

The *Tamari lattice*: a partial order on Catalan families



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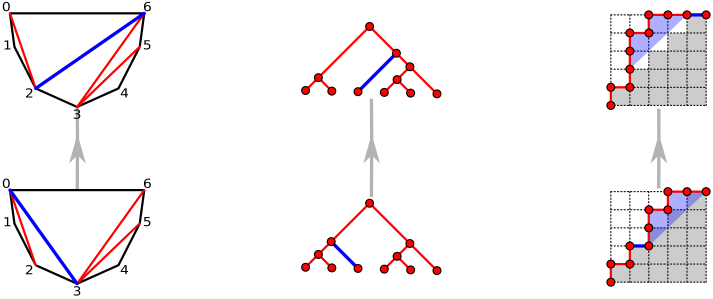
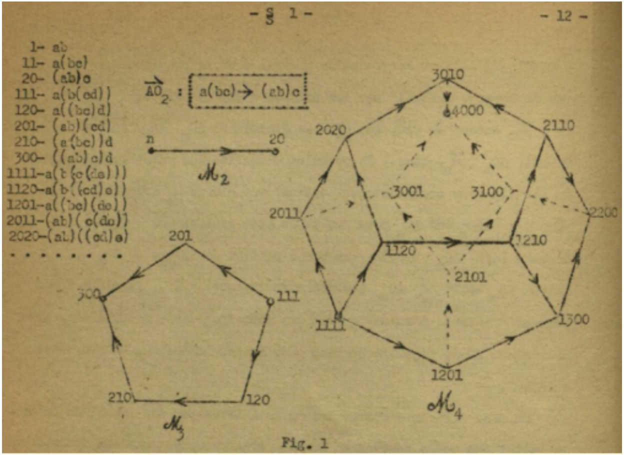


The Associahedron

The Tamari lattice can be realized by the graph of the *Associahedron*



[Dov Tamari, Monoïdes préordonnés et chaînes de Malcev, PhD thesis, Université de Paris, (1951)]

The Associahedron

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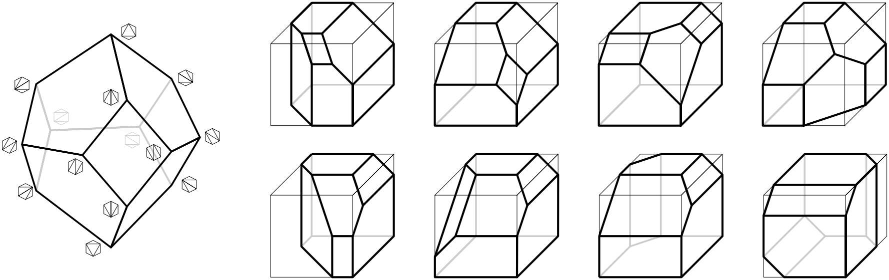
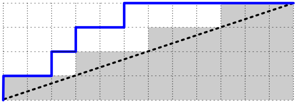


From [Ceballos, Santos, Ziegler, Many non-equivalent realizations of the Associahedron (2014)]

Stasheff ('63), Haiman ('84), Lee ('89), Gelfand-Kapranov-Zelevinsky ('94), Chapoton-Fomin-Zelevinsky ('02), Rote-Santos-Streinu ('03), Loday ('04), Hohlweg-Lange ('07), Postnikov ('09), Ceballos-Santos-Ziegler ('14)... among many others.

The m-Tamari lattice

Fuss-Catalan path: lattice path from $(0,0)$ to (mn, n) that stays weakly above the main diagonal.



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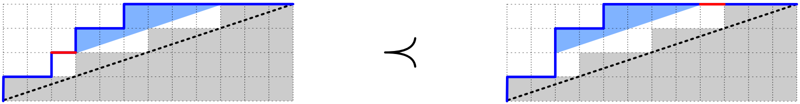
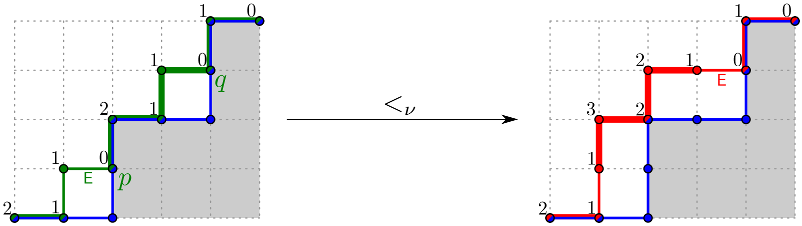


m-Tamari lattice: poset (actually a lattice) on Fuss-Catalan paths determined by this covering relation

[François Bergeron and Louis-François Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets, (2012)]

The ν-Tamari lattice

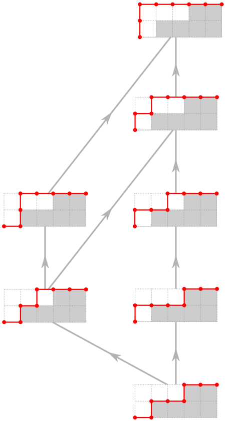
ν-Tamari lattice: poset (actually a lattice) on ν-paths determined by the following covering relation



ν-path: lattice path that stays weakly above ν.

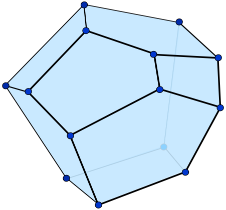
[Louis-François Préville-Ratelle and Xavier Viennot, An extension of Tamari lattices, (2014)]

The ν-Tamari lattice

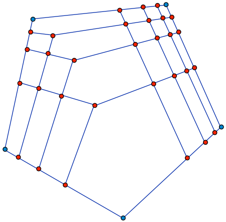


[Louis-François Préville-Ratelle and Xavier Viennot, An extension of Tamari lattices (2014)]

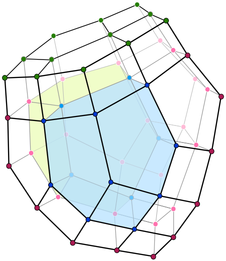
The $(\mathrm{NE}^m)^n$-Tamari lattice



1-Tamari $n = 4$



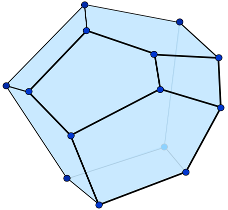
4-Tamari $n = 3$



2-Tamari $n = 4$

[François Bergeron. Combinatorics of r-Dyck paths, r-Parking functions, and the r-Tamari lattices (2012)]

The $(\mathrm{NE}^m)^n$-Tamari lattice



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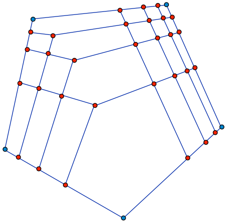
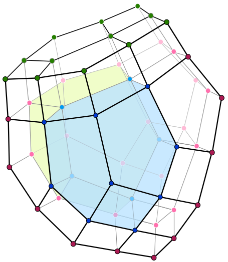


4-Tamari $n = 3$



2-Tamari $n = 4$

[François Bergeron. Combinatorics of r-Dyck paths, r-Parking functions, and the r-Tamari lattices (2012)]

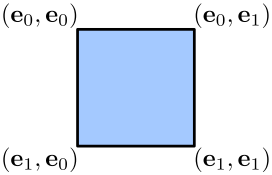
> There has also been recently a lot of work on geometric (or polytopal) realizations of the associahedron (the Tamari poset for $r = 1$). This leads to the natural question of describing similar constructions for all r-Tamari posets. Figure 6 suggests a tantalizing outlook on this.

A geometric realization of the ν-Associahedron

The associahedral triangulation

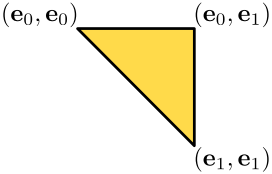
Consider the *product of two simplices*

$$\Delta_n \times \Delta_{\overline{n}} = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) : 0 \leq i, \bar{j} \leq n\right\}.$$



We want to triangulate (subdivide into simplicies) the sub-polytope

$$\mathcal{C}_n = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) : 0 \leq i \leq \bar{j} \leq n\right\}$$



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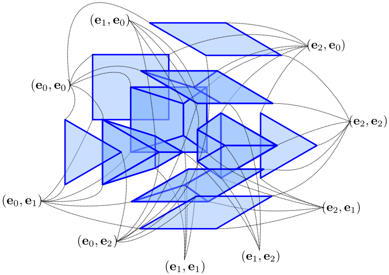

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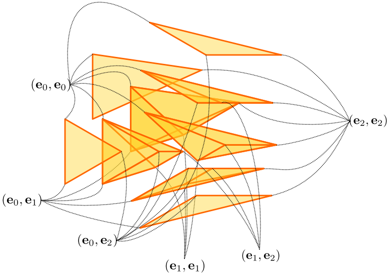
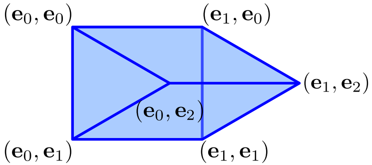


The associahedral triangulation



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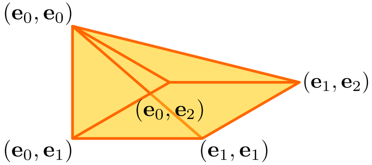
$$\Delta_n \times \Delta_{\overline{n}} = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) \colon 0 \leq i, \bar{j} \leq n\right\}.$$



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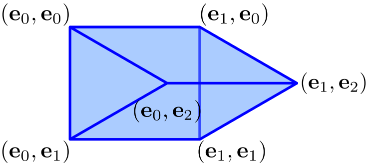

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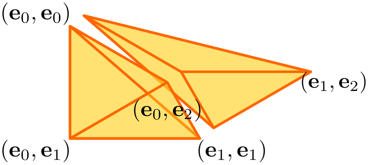
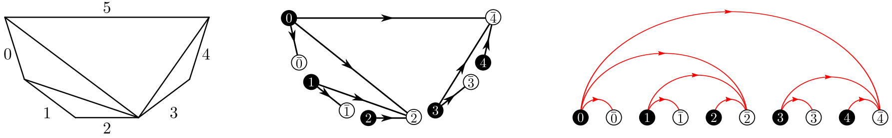


The associahedral triangulation

The cells: indexed by triangulations of an $(n+2)$-gon



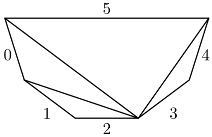
In this example, the cell is:

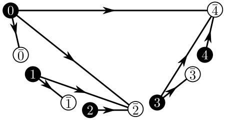
$$\mathrm{conv}\left\{(\mathbf{e}_0,\mathbf{e}_{\overline{0}}),(\mathbf{e}_0,\mathbf{e}_{\overline{2}}),(\mathbf{e}_0,\mathbf{e}_{\overline{4}}),(\mathbf{e}_1,\mathbf{e}_{\overline{1}}),\ldots,(\mathbf{e}_4,\mathbf{e}_{\overline{4}})\right\}$$

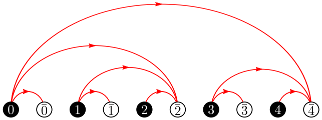
Alternatively: *bipartite non-crossing* trees on $[n]\sqcup[\overline{n}]$.

The associahedral triangulation

The cells: indexed by triangulations of an $(n+2)$-gon







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Alternatively: *bipartite non-crossing* trees on $[n] \sqcup [\overline{n}]$.

Fact

- *These cells triangulate the polytope $\mathcal{C}_n \subset \Delta_n \times \Delta_{\overline{n}}$*
- *This triangulation is dual to an associahedron.*
- *The triangulation is regular and flag.*

The associahedral triangulation

This triangulation has appeared under different disguises in many independent papers:

- As a triangulation of a root polytope
  [Gelfand, Graev, and Postnikov, Combinatorics of hypergeometric functions associated with positive roots (1997)]
- As a mixed subdivision of the Stanley – Pitman polytope
  [Stanley and Pitman, A polytope related to empirical distributions, plane trees, parking functions, and the associahedron (2002)]
- As a triangulation of a Gelfand-Tseltsin polytope
  [Petersen, Pylyavskyy, and Speyer, A non-crossing standard monomial theory (2010)]
- As a triangulation of an Order polytope
  [Santos, Stump, and Welker, Noncrossing sets and a Grassmann associahedron (2014)]
- ...

The $(I,\bar{J})$-triangulation

Faces of $\Delta_n \times \Delta_{\overline{n}}$ are of the form

$$\Delta_I \times \Delta_{\bar{J}} = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) : i \in I \text{ and } \bar{j} \in \bar{J}\right\}$$

where $I \subseteq [n]$ and $\bar{J} \subseteq [\overline{n}]$.

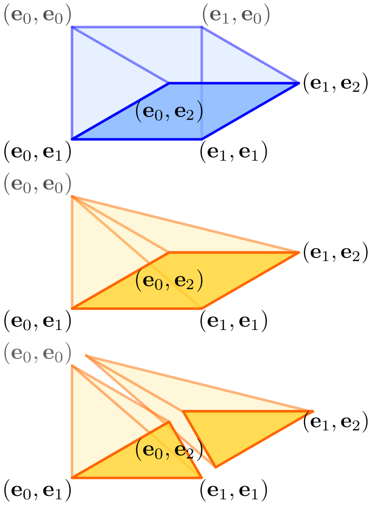
They induce faces of $\mathcal{C}_n$

$$\mathcal{C}_{I,\bar{J}} = \mathcal{C}_n \cap (\Delta_I \times \Delta_{\bar{J}})$$

The restriction of the associahedral triangulation to the face

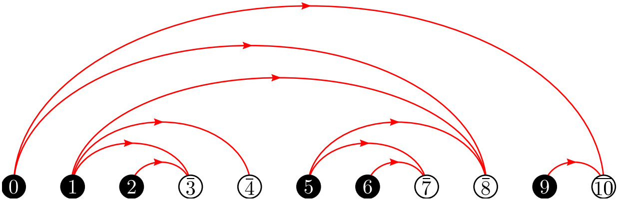
$$\mathcal{C}_{I,\bar{J}} = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) : i \in I \text{ and } \bar{j} \in \bar{J}, i \leq \bar{j}\right\}$$

is called the $(I,\bar{J})$-triangulation.



The $(I, \bar{J})$-triangulation

The cells of this restricted triangulation are indexed by *$(I, \bar{J})$-trees*
(bipartite non-crossing trees with support $I \cup \bar{J}$)

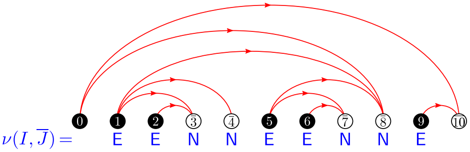


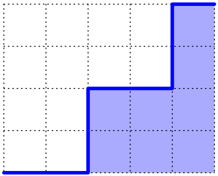
In this example,

$$I = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{6}, \mathbf{9}\} \qquad \bar{J} = \{\overline{\mathbf{3}}, \overline{\mathbf{4}}, \overline{\mathbf{7}}, \overline{\mathbf{8}}, \overline{\mathbf{10}}\}$$

The $(I, \bar{J})$-triangulation

Given such a tree T we associate two paths $\nu(I, \bar{J})$ and $\rho(T)$:





$\nu(I, \bar{J})$ replaces black and white balls by east and north steps respectively.

The $(I, \bar{J})$-triangulation

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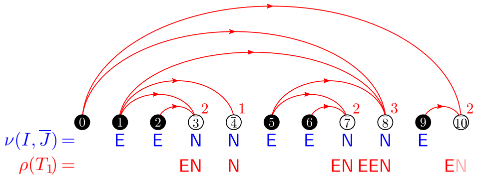
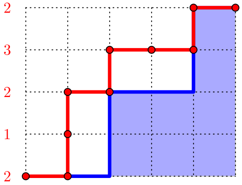




$\nu(I, \bar{J})$ replaces black and white balls by east and north steps respectively.
$\rho(T)$ counts the in-degrees of the white balls.

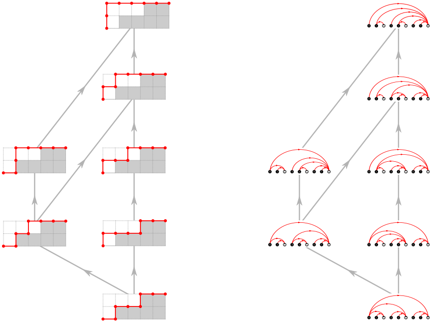
Note: the path $\rho(T)$ is weakly above ν.

The lattice of flips of $(I, \bar{J})$-trees

Proposition (CPS)

Let I, J be a partition of $[n]$ with $0 \in I$ and $n \in J$, and $\nu = \nu(I, \bar{J})$.

- *ρ is a bijection from $(I, \bar{J})$-trees to ν-paths.*
- *flips of $(I, \bar{J})$-trees correspond to ν-Tamari covering relations.*



The ν-Tamari lattice as the dual of a triangulation

Theorem (CPS)

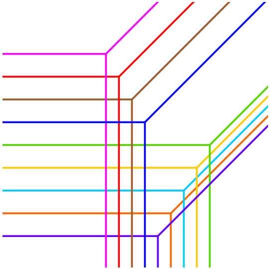
The ν-Tamari lattice $\mathrm{Tam}(\nu)$ *can be realized geometrically as the dual of a regular triangulation of a sub-polytope of* $\Delta_a \times \Delta_b$.

Tropical geometry

Tropical semiring
$(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$
where $x \oplus y := \max(x, y)$ and $x \odot y = x + y$

$\rightsquigarrow$

Tropical Geometry
Tropical polynomials, tropical curves,
tropical hyperplanes, tropical lines ...



An arrangement of tropical lines

Tropical geometry

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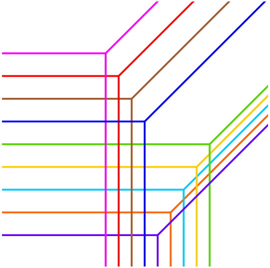
Tropical polynomials, tropical curves, tropical hyperplanes, tropical lines . . .



An arrangement of tropical lines

Theorem (Develin-Sturmfels ’04)

$$\left\{\begin{matrix}\textit{regular triangulations}\\ \textit{of } \Delta_n \times \Delta_{\overline{m}}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{(combintorial types of) generic arrangements of}\\ m \textit{ tropical hyperplanes in } \mathbb{TP}^{n-1}\end{matrix}\right\}$$

Tropical geometry

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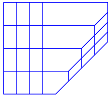

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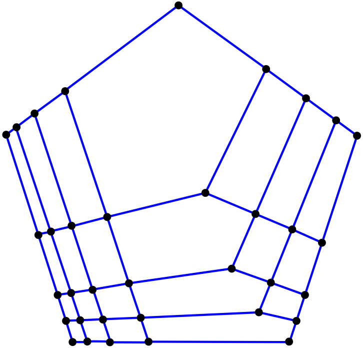
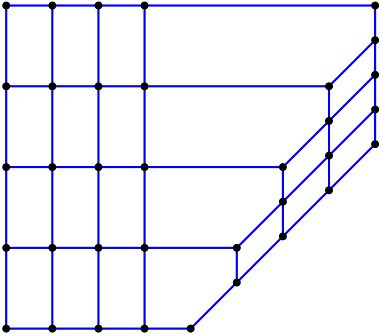
A realization of the ν-Tamari lattice

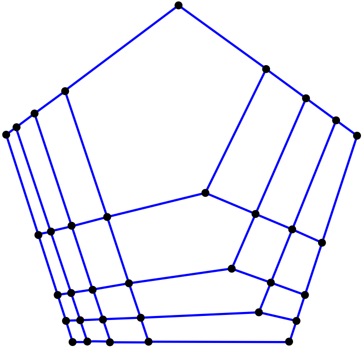


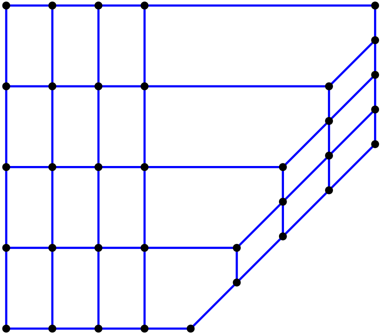


A realization of the ν-Tamari lattice

Theorem (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ *is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.*



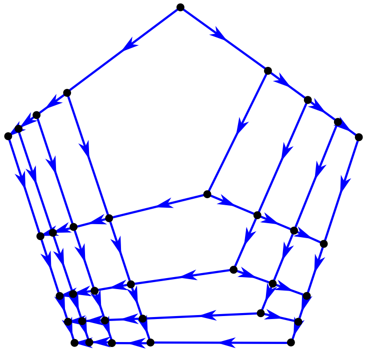


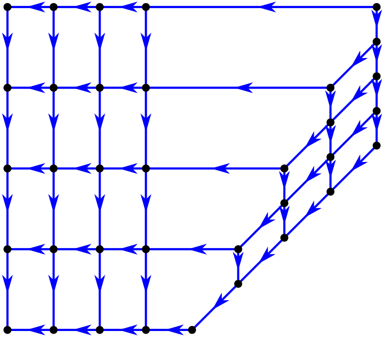
The 4-Tamari lattice for $n = 3$.

A realization of the ν-Tamari lattice

Theorem (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ *is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.*





The 4-Tamari lattice for $n = 3$.

A realization of the ν-Tamari lattice

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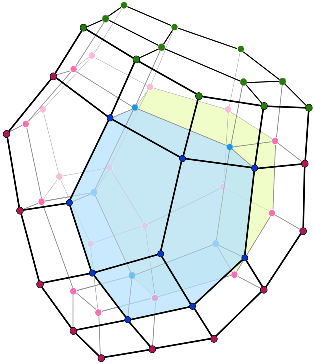
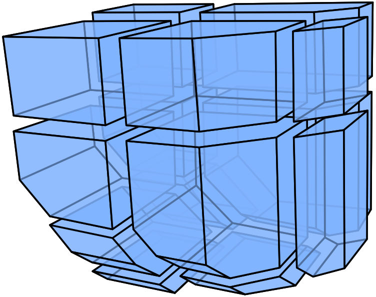


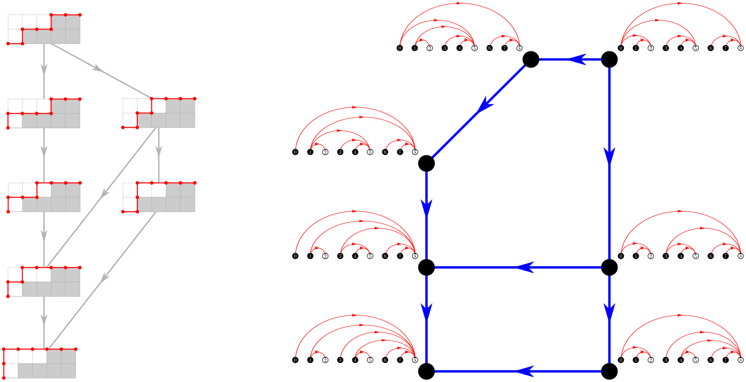


The 2-Tamari lattice for $n = 4$.

A realization of the ν-Tamari lattice

Theorem (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.

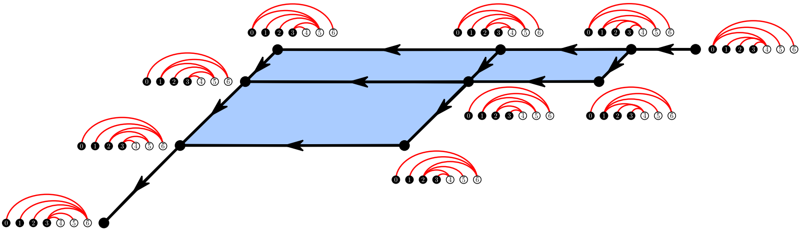


The rational Tamari lattice $\mathrm{Tam}(3,5)$.

A realization of the ν-Tamari lattice

Theorem (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ *is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.*



The $(\{0,1,2,3\},\{\overline{4},\overline{5},\overline{6}\})$-Tamari lattice.

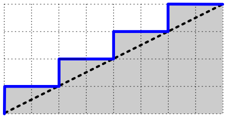
A realization of the ν-Tamari lattice

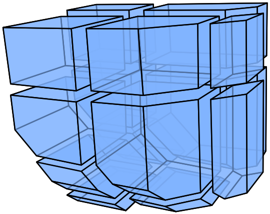
Theorem (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.

Theorem

The support of Asso_ν is convex only if ν does not have two (non-inicial) consecutive north steps.
In this case, Asso_ν is a subdivision of a classical associahedron into Cartesian products of associahedra.

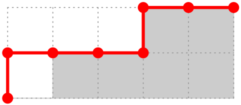


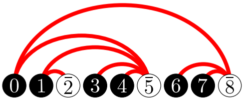


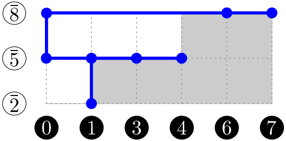
Combinatorial connections

ν-trees

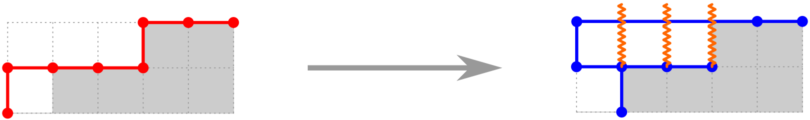
ν-trees







ν-trees



ν-trees

Definition

A *ν-tree* is a maximal collection of pairwise ν-compatible points inside the partition bounded by ν.

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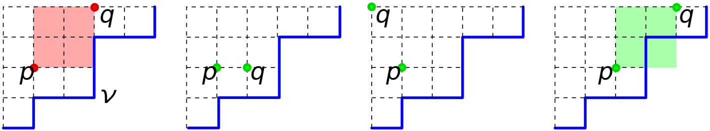
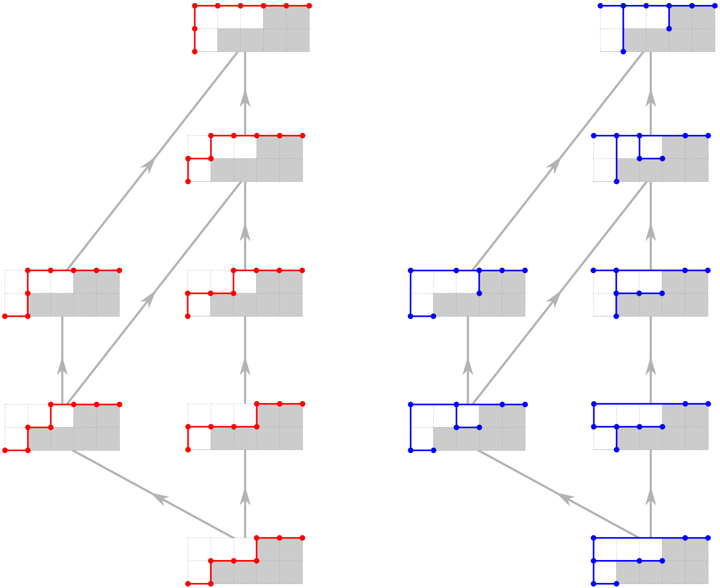


ν-incompatible ν-compatible ν-compatible ν-compatible

The rotation lattice of ν-trees

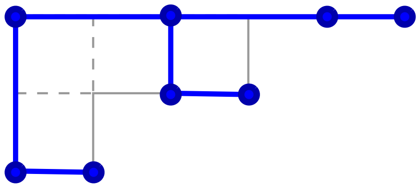


Other incarnations of ν-trees

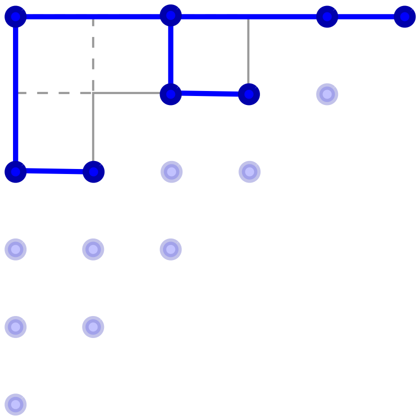
ν-trees are also:

- the $k = 1$ case of *k-north-east fillings* $\longrightarrow$ relation to *pipe dreams* and *subword complexes*
  [Serrano and Stump, Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials (2012)]
- the same as *non-crossing tree-like tableaux*, the Catalan subfamily of tree-like tableaux, in bijection with *permutation* and *alternative tableaux*.
  [Aval, Boussicault, and Nadeau, Tree-like tableaux (2013)]

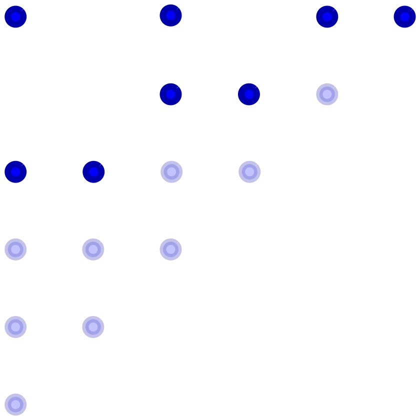
Pipe dreams



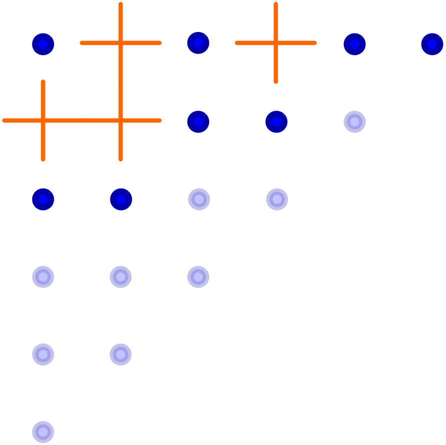
Pipe dreams



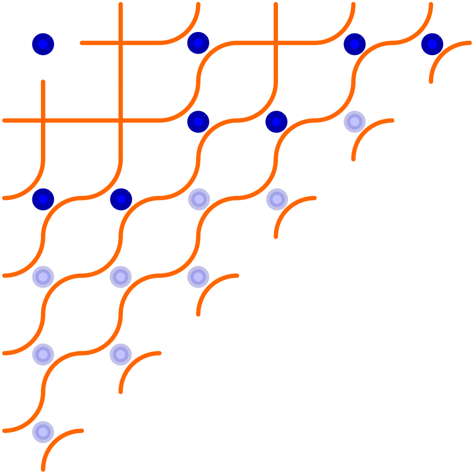
Pipe dreams



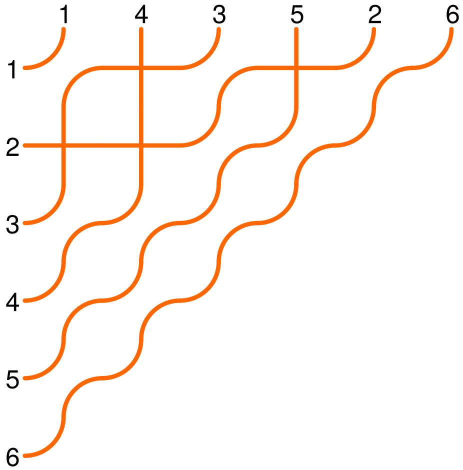
Pipe dreams



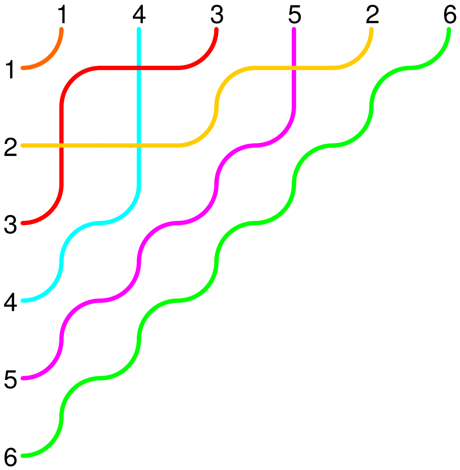
Pipe dreams



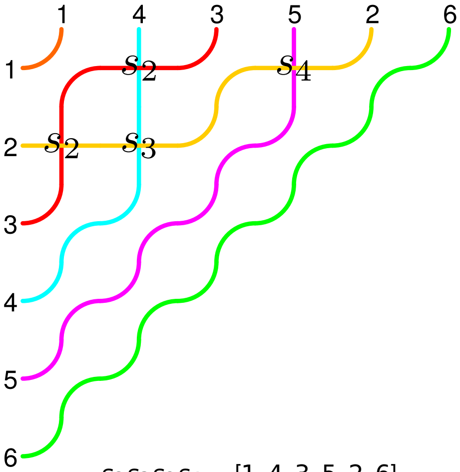
Pipe dreams



Pipe dreams



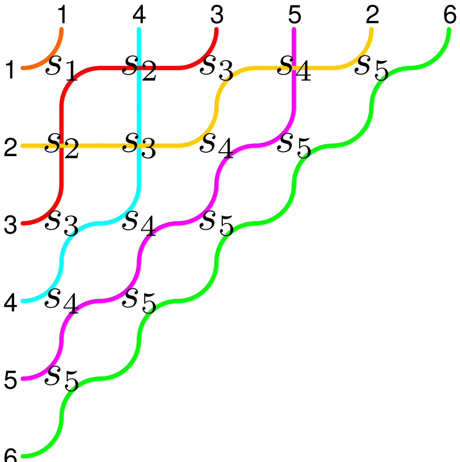
Pipe dreams



$$s_2 s_3 s_2 s_4 = [1, 4, 3, 5, 2, 6]$$

$$s_i = (i, i+1)$$

Pipe dreams



$$s_2 s_3 s_2 s_4 = [1, 4, 3, 5, 2, 6]$$

$$s_i = (i, i+1)$$

Subword complexes

Definition

Let $Q = (q_1, \ldots, q_m) \in \{s_1, \ldots, s_n\}^m$ and $\pi \in \mathfrak{S}_{n+1}$ be a permutation. The *subword complex* is a *simplicial complex* whose faces are given by subsets $I \subset \{1, 2, \ldots, m\}$, such that the subword of Q with positions at $[m] \smallsetminus I$ contains a reduced expression of π.
[Knutson and Miller, Subword complexes in Coxeter groups (2004)]

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_5	s_4	s_3	s_2	s_1	s_5	s_4	s_3	s_2	s_5	s_4	s_3	s_5	s_4	s_5

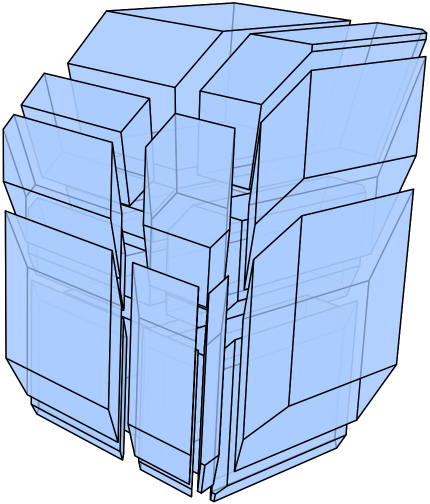
$$\rightsquigarrow$$

$$\{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 15\}$$

Theorem

The Hasse diagram of the ν-Tamari lattice is isomorphic to the facet adjacency graph of a subword complex.

Merci beaucoup! Moltes gràcies!



Thank you!

Type B: ν-cyclohedra