Eulerian polynomials and generalizations

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LIGM

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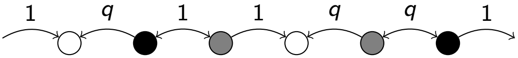
- 

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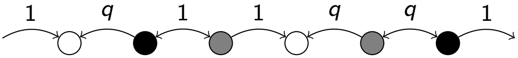
- 

- $$\alpha_n(t,q) = \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{seg}(\sigma)}$$

Descents of a permutation

A position i of a permutation σ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\mathrm{des}(\sigma)$ the number of descents of σ.
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Eulerian numbers

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1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
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$$A(n, k) = (n - k)A(n - 1, k - 1) + (k + 1)A(n - 1, k)$$

Eulerian polynomials

For any $n \geq 0$, define the Eulerian polynomials as:

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{k=0}^{n-1} A(n,k) t^k.$$

For example, we have

$$\begin{aligned} A_3(t) &= 1 + 4t + t^2; \\ A_4(t) &= 1 + 11t + 11t^2 + t^3. \end{aligned}$$

Some results about Eulerian numbers and polynomials

- Worpitzky's identity: For any positive integers n and k,

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How to prove these ?

Generating function

Define the exponential generating function for the Eulerian polynomials as

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First proof

Using the recurrence of the Eulerian numbers, one proves that G satisfies the following differential equation

$$(1-tx)\frac{\partial}{\partial x}G(t,x) - t(1-t)\frac{\partial}{\partial t}G(t,x) - G(t,x) = 0.$$

Then prove that (1) is a solution.

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$$\mathbf{G}_\sigma \cdot \mathbf{G}_\tau = \sum_{\mu \in \sigma * \tau} \mathbf{G}_\mu,$$

where the permutations in the sum are in $\mathfrak{S}_{n+p}$. In order to have ϕ a morphism we need exactly $\binom{n+p}{n}$ elements in the sum.

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There are two possibilities, we consider the one where we concatenate σ and τ and consider all the possibilities to create a permutation. For example,

$$\mathbf{G}_{312} \cdot \mathbf{G}_{21} = \mathbf{G}_{31254} + \mathbf{G}_{41253} + \mathbf{G}_{41352} + \mathbf{G}_{42351} + \mathbf{G}_{51243} + \cdots + \mathbf{G}_{53421}$$

Noncommutative analog

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Using ϕ we have $\phi(\mathcal{A}_n(t)) = tA_n(t)\dfrac{x^n}{n!}.$

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And more generally,

$$\mathcal{A}_n = \sum_{I \models n} t^{\ell(I)}\mathbf{R}_I$$

where **R** is the ribbon basis of the noncommutative symmetric functions algebra (**Sym**) and the sum goes over all composition of size n (sequences of integers of sum n).

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$$\mathbf{R}_I = \sum_{I \succeq J} (-1)^{\ell(I)-\ell(J)} \mathbf{S}^J.$$

For example,

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Then,

$$\mathcal{A}_3(t) = t(1-t)^2\mathbf{S}_3 + t^2(1-t)\left(\mathbf{S}^{12} + \mathbf{S}^{21}\right) + t^3 S^{111}.$$

We have

$$\mathcal{A}_n = \sum_{I \models n} t^{\ell(I)}(1-t)^{n-\ell(I)} \mathbf{S}^I;$$

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To apply ϕ, we use the fact that $\mathbf{S}$ is a multiplicative basis. For example,

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Moreover, $\phi(S_k) = \frac{x^k}{k!}$ so $\phi\left(\sum_{\substack{I \models n \\ \ell(I)=r}} \mathbf{S}^I\right) = r!S(n,r)\frac{x^n}{n!}$.

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Moreover, $\phi\left(S_k\right) = \dfrac{x^k}{k!}$ so $\phi\left(\sum_{\substack{I \models n\\ \ell(I)=r}} \mathbf{S}^I\right) = r!S(n,r)\dfrac{x^n}{n!}$. Then,

$$tA_n(t) = \sum_{r=1}^{n} t^r(1-t)^{n-r} r! S(n,r).$$

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$$\sum_{n\geq 0} \frac{\mathcal{A}_n}{(1-t)^n} = \sum_{r\geq 0} \left(\frac{t}{1-t}\right)^r (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \cdots)^r.$$

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Then applying ϕ to the previous equation we obtain

$$1 + \sum_{n \geq 0} \frac{tA_n(t)}{(1-t)^n} \frac{x^n}{n!} = \frac{1-t}{1-te^x}$$

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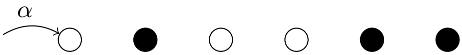

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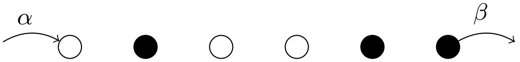

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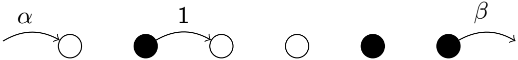

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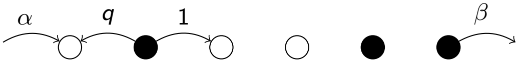

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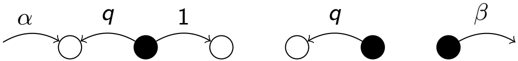

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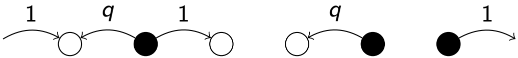

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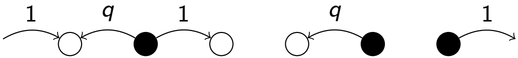

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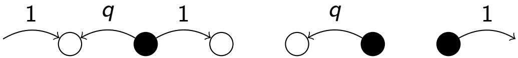

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The steady-state probabilities of the states of the ASEP can be described combinatorialy using some statistics on permutations.

Recoils of a permutation

A recoil of a permutation σ is a value σ_i such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$.

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Theorem

Let X be a state of size n of the ASEP. The steady-state probability of X is proportional to the number of permutations of $\mathfrak{S}_{n+1}$ having their recoils in the same position as the empty spots of X

In fact, sorting the permutations according to their position of recoils and position of descents allow us to compute transitions matrices between to bases of **Sym**.

PRec \ Des	$\emptyset$	$\{3\}$	$\{2\}$	$\{2,3\}$	$\{1\}$	$\{1,3\}$	$\{1,2\}$	$\{1,2,3\}$
$\emptyset$	1234							
$\{3\}$		1243, 1423 4123	1342 3412		2341	2413		
$\{2\}$			1324 3124		2314			
$\{2,3\}$			3142	1432, 4132 4312		2431 4231	3241	
$\{1\}$					2134			
$\{1,3\}$						2143 4213	3421	
$\{1,2\}$							3214	
$\{1,2,3\}$								4321

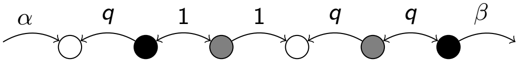
Theorem (Hivert, Novelli, Tevlin, Thibon, 2009)

Let G^n be the transition matrix between the ribbon basis of **Sym** *and the fundamental one in degree n. Let S and T be two subsets of $[n-1]$, then*

$$G^n_{S,T} = \#\{\sigma \in \mathfrak{S}_{n+1} |\, \mathrm{Des}(\sigma) = S, \mathrm{PRec}(\sigma) = T\}.$$

2-ASEP

The 2-ASEP is a generalization of the ASEP with two kinds of particles.



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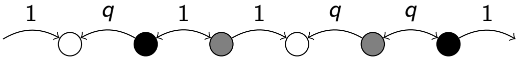

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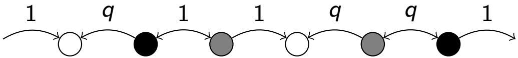

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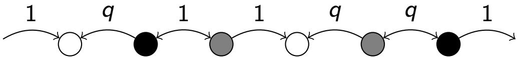

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In 2016, O. Mandelshtam and X. Viennot defined a statistic on "Assemblées of permutations" to describe the combinatorics of the 2-ASEP at $q = 1$. Where an assemblée of permutation is a permutation σ segmented in blocks where the order of the blocks is not important.
For example, $\sigma = [251][84][637] = [84][251][637]$.

Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathfrak{P}_n$ the set of segmented permutations of size n. For example, $\sigma = 52|7138|46 \in \mathfrak{P}_8$.

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A recoil of σ is a value σ_i such that i is not a segmentation and $1 + \sigma_i$ is to the left of σ_i. Denote by $\mathrm{PRec}(\sigma)$ the set of the position of recoils minus one.

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Theorem (Corteel, N., 2018+)

Let X be a state of the 2-ASEP of size n. The steady-state probability of X is proportional to the number of segmented permutations $\sigma \in \mathfrak{P}_{n+1}$ such that $\mathrm{Seg}(\sigma)$ *corresponds to the position of the particles of type 2 and* $\mathrm{PRec}(\sigma)$ *corresponds to the position of the empty spots.*

Eulerian numbers on segmented permutations

We define the following numbers:

$$T(n,k) = \#\{\sigma \in \mathfrak{P}_n | \operatorname{des}(\sigma) = k\}$$

$n \backslash k$	0	1	2	3	4
0	1				
1	1				
2	3	1			
3	13	10	1		
4	75	91	21	1	
5	541	896	426	56	1

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We also have

$$T(n,n-k-1) = \#\{\sigma \in \mathfrak{P}_n | \operatorname{des}(\sigma) + \operatorname{seg}(\sigma) = k\}.$$

Generalized Eulerian numbers with two parameters

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$j \backslash i$	0	1
0	1	1
1	2	

$n = 3:$

$j \backslash i$	0	1	2
0	1	4	1
1	6	6	
2	6		

$n = 4:$

$j \backslash i$	0	1	2	3
0	1	11	11	1
1	14	44	14	
2	36	36		
3	24			

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For any n, the numbers on each first row are the usual Eulerian numbers and the numbers on each first column are the $j!S(n,j)$. We have

$$\begin{aligned} K(n,i,j) = (i+j+1)\Big[K(n-1,i,j) + K(n-1,i,j-1)\Big] \\ +(n-i-j)\Big[K(n-1,i-1,j) + K(n-1,i-1,j-1)\Big]. \end{aligned}$$

Generalized Eulerian polynomials

Define our polynomials as

$$\alpha_n(t,q) = \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{seg}(\sigma)}.$$

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Some specialization of the variables give the following properties

$$\begin{cases} \alpha_n(t,0) &= A_n(t) \\ \alpha_n(0,q) &= B_n(q) \\ \alpha_n(1,1) &= 2^{n-1} n! \\ \alpha_n(-1,1) &= 2^{n-1} \\ \alpha_n(2,1) &= A050352 \\ \alpha_n(2,2) &= A050351 \end{cases}$$

Generating Function

We define the generating function of the generalized Eulerian polynomials as follows:

$$G(t,q,x) = \sum_{n\geq 0} \alpha_n(t,q)\frac{x^n}{n!}.$$

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$$(tq-2q-1)G(t,q,x) + (1-tqx-tx)\frac{\partial}{\partial x}G(t,q,x) - (t-t^2)(q+1)\frac{\partial}{\partial t}G(t,q,x) - (1-t)(q^2+q)\frac{\partial}{\partial q}G(t,q,x) = -2q+tq.$$

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Theorem (N., 2018+)

We have the following expression of the generating function:

$$G(t,q,x) = 1 + \frac{e^{x(1-t)}-1}{1+q-(t+q)e^{x(1-t)}}.$$

Some properties of the generalized Eulerian polynomials

- Worpitzky's identity: for any positive integers r, k, and n,

$$\binom{k+r-1}{r}\Delta^{r+1}((k-1)^n) = \sum_{i=0}^{k-1}\binom{n+k-i}{n-1}K(n,i,r),$$

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Merci de votre attention !