Tree substitutions and Rauzy fractals

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(joint work with Thierry Coulbois)

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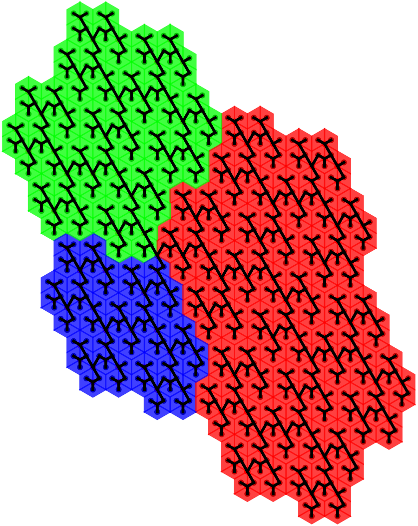
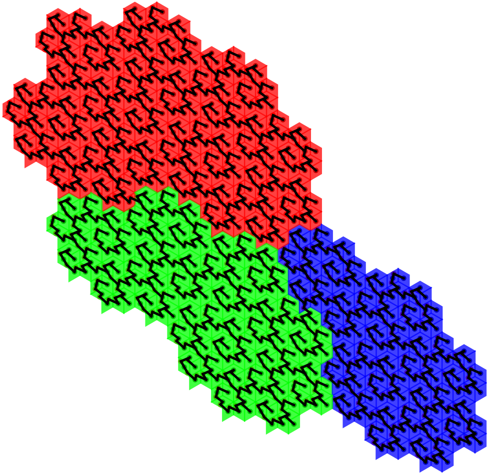
Tree-bonacci

$\sigma : a \mapsto ab,\ b \mapsto ac,\ c \mapsto a$



Trees and Rauzy fractals

$\sigma : a \mapsto ac,\ b \mapsto ab,\ c \mapsto b$



Trees and Rauzy fractals

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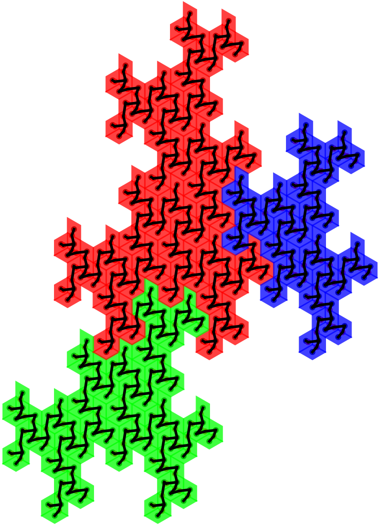
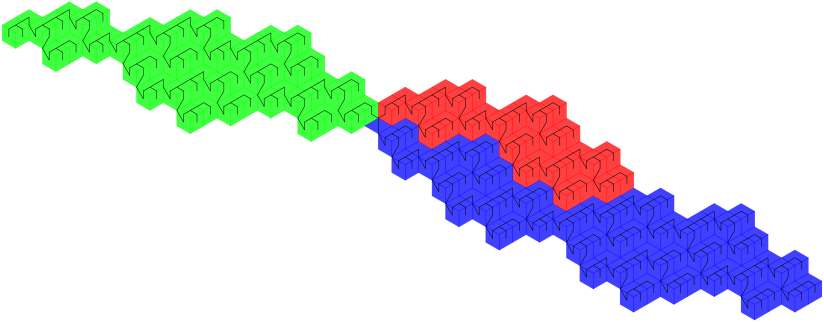


Trees and Rauzy fractals

$\sigma : a \mapsto abc,\ b \mapsto bcabc,\ c \mapsto cbcabc$



Substitutions

Tribonacci substitution

$$\sigma : a \mapsto ab,\ b \mapsto ac,\ c \mapsto a$$
$$\sigma(a) = ab$$

Substitutions

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$$\sigma : a \mapsto ab,\ b \mapsto ac,\ c \mapsto a$$

$$\sigma^2(a) = abac$$

Substitutions

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Substitutions

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$$M_{\sigma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f(x) = x^3 - x^2 - x - 1$$

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The Tribonacci substitution is an irreducible unit Pisot substitution.

Substitutive subshifts

(X_σ, S)

$X_\sigma =$ set of bi-infinite words in $A^{\mathbb{Z}}$ whose factors are in the language

$$L_\sigma = \{w \in A^* : w \text{ is a factor of } \sigma^n(i) \text{ for some } i \in A, n \in \mathbb{N}\}.$$

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For primitive σ, (X_σ, S) is:

- minimal
- uniquely ergodic
- zero entropy

Pisot substitutions

Tribonacci substitution

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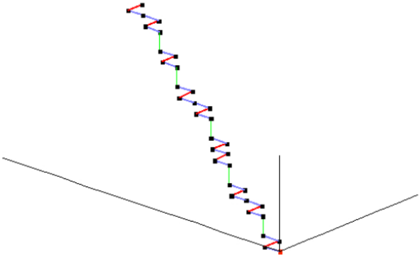
$$\sigma^\infty(a) = abacabaabacababacabaabac \cdots$$

Abelianization of $\sigma^\infty(a)$



Pisot $\Rightarrow$ Balance: stepped line stays at bounded distance from $\mathbb{R}\mathbf{u}$.

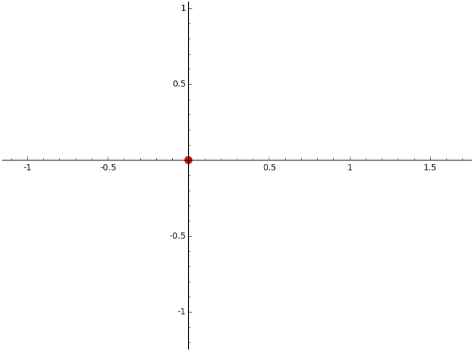
The Rauzy fractal

Space decomposition: $\mathbb{R}^3 = \mathbb{R}\mathbf{u} \oplus H$ ($H =$ contracting space of M_σ).

Fixed point: $\sigma^\infty(a) = \epsilon abacabaabacababacabaabac\cdots$.
Vertex of stepped line: $\ell(p) = \mathbf{0}$

Rauzy fractal

$\mathcal{R} = \bigcup_{i\in\mathcal{A}} \mathcal{R}(i)$ where $\mathcal{R}(i) = \overline{\{\pi(\ell(p)) : pi \text{ prefix of } \sigma^\infty(a)\}} \subset H$.



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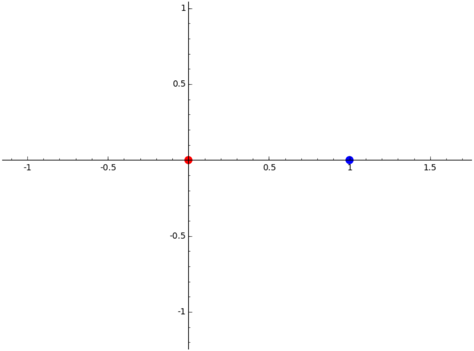

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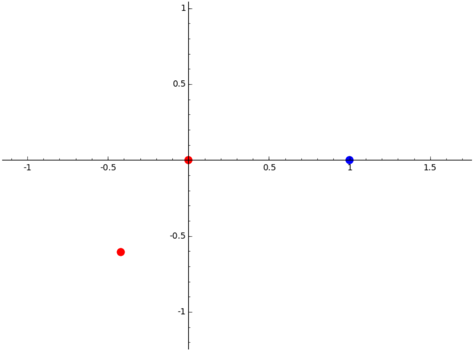

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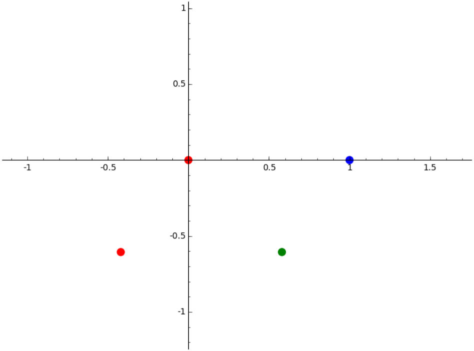

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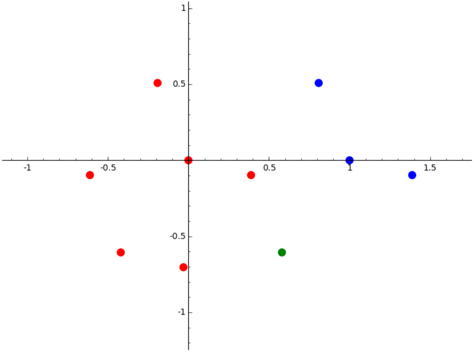

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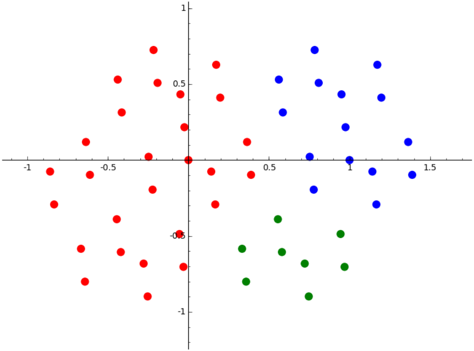

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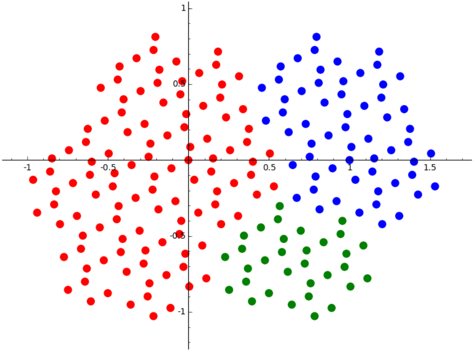

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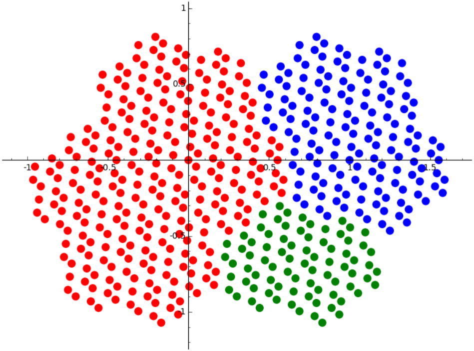

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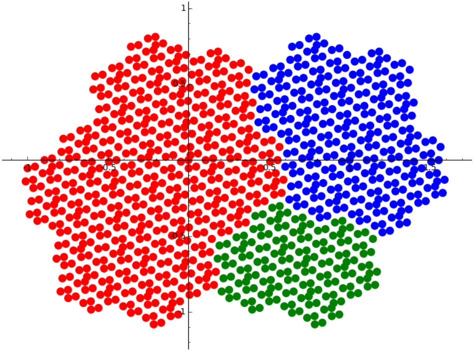

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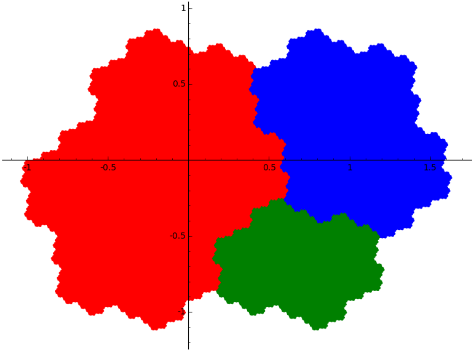
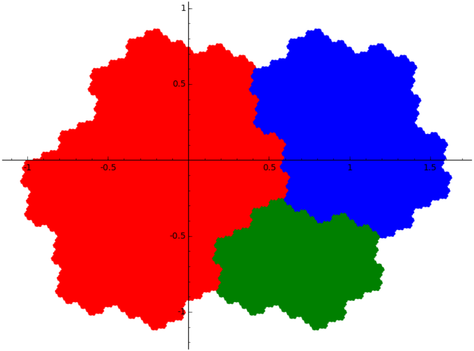


Domain exchange



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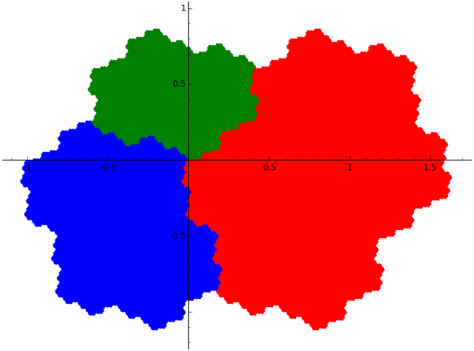

Domain exchange $E : \mathcal{R}(i) \mapsto \mathcal{R}(i) + \pi(\mathbf{e}_i)$.

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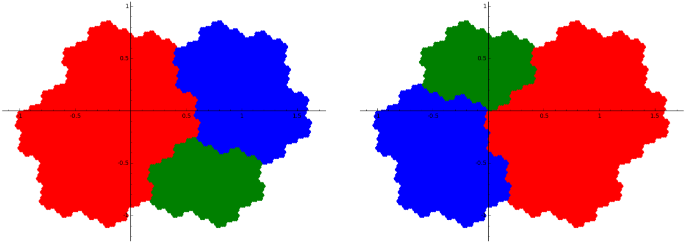


Domain exchange $E : \mathcal{R}(i) \mapsto \mathcal{R}(i) + \pi(\mathbf{e}_i)$.

Theorem (Arnoux-Ito 01)

For any irreducible unimodular Pisot substitution satisfying the strong coincidence condition (X_σ, S) is measurably conjugate to $(\mathcal{R}, E)$.

$$\begin{tikzcd} X_\sigma \arrow[r, "S"] \arrow[d, "\varphi"'] & X_\sigma \arrow[d, "\varphi"] \\ \mathcal{R} \arrow[r, "E"] & \mathcal{R} \end{tikzcd}$$

Tilings

Do Rauzy fractals induce a periodic tiling of the space where they live?

Tilings

Tiling by Rauzy fractals $\Longrightarrow \mathcal{R}$ fundamental domain for $\mathbb{T}^{d-1}$.
In this case, the domain exchange E projects onto the torus to a translation T.

Pisot conjecture

A coding inverse problem:

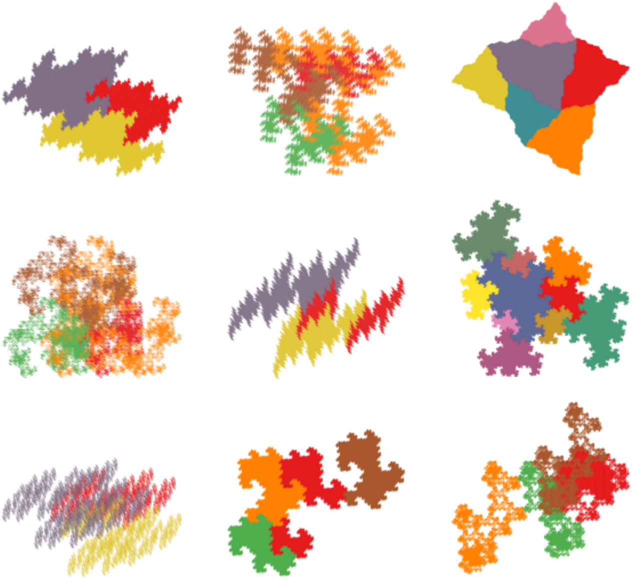
Pisot conjecture

Let σ be an irreducible unimodular Pisot substitution. Then (X_σ, S) has pure discrete spectrum, or equivalently it is metrically isomorphic to a translation on a torus $\mathbb{T}^{d-1}$.
Equivalently each $\omega \in X_\sigma$ is a natural coding of the toral translation T w.r.t. partition $\{\mathcal{R}(i) : i \in \mathcal{A}\}$.

$$\begin{tikzcd}
X_\sigma \arrow[r, "\varphi"] \arrow[d, "S"'] & \mathcal{R} \arrow[r, "\cong ?"] \arrow[d, "E"] & \mathbb{T}^{d-1} \arrow[d, "T"] \\
X_\sigma \arrow[r, "\varphi"] & \mathcal{R} \arrow[r, "\cong ?"] & \mathbb{T}^{d-1}
\end{tikzcd}$$

Rauzy fractals

Drawing with SAGE:



Desubstitution

Desubstitution (Mossé 92)

Each $w \in X_\sigma$ can be written as

$$w = S^{|p|}(\sigma(w')), \quad \text{for some } w' \in X_\sigma, p \in A^*$$

Prefix-suffix graph: $w_0 \xleftarrow{p,s} w_0'$, i.e. $\sigma(w_0') = pw_0s \in A^* \times A \times A^*$

Iterating we have that each $w \in X_\sigma$ has an infinite desubstitution path:

$$\Gamma : X_\sigma \to \mathcal{P}, \quad w \mapsto \gamma = w_0 \xleftarrow{p_0,s_0} w_0' \xleftarrow{p_1,s_1} w_0'' \xleftarrow{p_2,s_2} \cdots$$

Γ is onto, continuous and 1-to-1 except on the set of periodic points of σ.

Desubstitution

Example (Tribonacci)

$$\cdots cabaabacababacabaabacaba \,.\, cabaabacababacabaabacaba \cdots$$

Desubstitution

Example (Tribonacci)

$$\cdots|ac|ab|ab|ac|ab|a|ab|ac|ab|a\,.\,\mathbf{c}|ab|a|ab|ac|ab|ab|ac|ab|a|ab|ac\cdots$$

Desubstitution

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$$\cdots|ac|ab|ab|ac|ab|a|ab|ac|ab|a\,.\,\mathbf{c}|ab|a|ab|ac|ab|ab|ac|ab|a|ab|ac\cdots$$
$$\cdots b \quad a\,a \quad b\,a \quad c\,a \quad b\,a \quad .\mathbf{b}\,a \quad c\,a \quad b\,a \quad a\,b \quad a\,c \quad a\,b\cdots$$

Desubstitution

Example (Tribonacci)

$$\cdots|ac|ab|ab|ac|ab|a|ab|ac|ab|a\,.\,\mathbf{c}|ab|a|ab|ac|ab|ab|ac|ab|a|ab|ac\ \cdots$$

$$\cdots\ b\quad|a|a\quad b|a\quad c|a\quad b|a\quad .\,\mathbf{b}|a\quad c|a\quad b|a\quad |a\ b\quad |a\ c\quad |a\ b\ |\cdots$$

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$$\cdots\ a\quad c\ a\qquad b\quad a\ c\quad .\,\mathbf{a}\ b\quad a\quad c\quad a\quad b\quad a\quad \cdots$$

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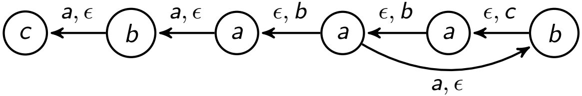
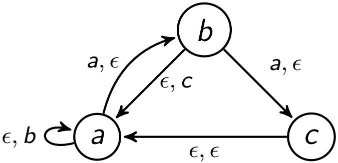
$$\cdots a\quad c\ a\quad b\quad a\ c\quad .\,\mathbf{a}\ b\quad a\quad c\quad a\quad b\quad a\quad \cdots$$

The starting word is encoded by the path



in the prefix-suffix graph



Self-similarity

Cylinders: $[i] = \{w \in X_\sigma : w_0 = i\}$, $i \in A$.

$$[i] = \bigcup_{i \xleftarrow{p,s} j} S^{|p|}\sigma([j])$$

The map φ

$$\varphi : X_\sigma \to \mathcal{R}, \quad w \mapsto \sum_{k \geq 0} \pi(M_\sigma^k(\ell(p)))$$

$$\varphi(Sw) = \varphi(w) + \pi(\ell(w_0))$$

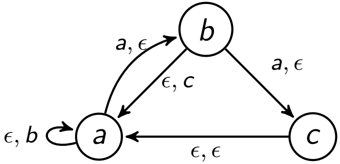
$$\varphi(\sigma(w)) = M_\sigma \varphi(w)$$

Graph-iterated function system (GIFS):

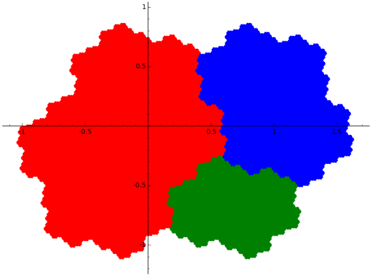
$$\mathcal{R}(i) = \bigcup_{i \xleftarrow{p,s} j} M_\sigma \mathcal{R}(j) + \pi(\ell(p))$$

Self-similarity

Prefix-suffix graph:



$$\mathcal{R}(a) = M_\sigma\,\mathcal{R}(a) \cup M_\sigma\,\mathcal{R}(b) \cup M_\sigma\,\mathcal{R}(c)$$
$$\mathcal{R}(b) = M_\sigma\,\mathcal{R}(a) + \pi(\mathbf{e}_1)$$
$$\mathcal{R}(c) = M_\sigma \mathcal{R}(b) + \pi(\mathbf{e}_1)$$



Singular words

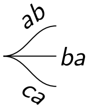
Tribonacci substitution

$$\begin{array}{ccccccc} a & \mapsto & ab & \mapsto & abac & \mapsto & abacaba \\ b & \mapsto & ac & \mapsto & aba & \mapsto & abacab \\ c & \mapsto & a & \mapsto & ab & \mapsto & abac \end{array}$$

$$\left.\begin{array}{l} w_a : \ \sigma^{3\infty}(a) = \cdots abacababacaba \\ w_b : \ \sigma^{3\infty}(b) = \cdots abacabaabacab \\ w_c : \ \sigma^{3\infty}(c) = \cdots ababacabaabac \end{array}\right\} \cdot abacabaabacabab \cdots = \sigma^{3\infty}(a)$$

Singular words (continued)

One periodic Nielsen path of period 3:



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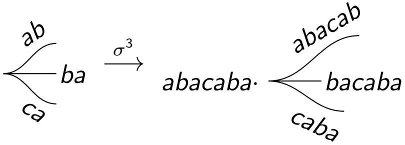

Singular words (continued)

One periodic Nielsen path of period 3:

$$\left\{\begin{matrix} ab \\ ba \\ ca \end{matrix}\right. \xrightarrow{\sigma^3} abacaba\cdot \left\{\begin{matrix} abacab \\ bacaba \\ caba \end{matrix}\right.$$

Fixed points of $i_{abacaba} \circ \sigma^3 = \widetilde{\sigma}^3$:

$$\widetilde{\sigma}^{3\infty}(a) = \cdots bacabaabacaba\cdot \begin{cases} abacababacaba\cdots = \widetilde{\sigma}^{3\infty}(a) & : w'_a \\ bacabaabacaba\cdots = \widetilde{\sigma}^{3\infty}(b) & : w'_b \\ cabaabacababa\cdots = \widetilde{\sigma}^{3\infty}(c) & : w'_c \end{cases}$$

(where $i_w(x) = w^{-1}xw$)

Index

Count the singular words:

$$w_a : \ \sigma^{3\infty}(a) = \cdots abacababacaba$$
$$w_b : \ \sigma^{3\infty}(b) = \cdots abacabaabacab$$
$$w_c : \ \sigma^{3\infty}(c) = \cdots ababacabaabac$$
$$\cdot abacabaabacabab \cdots = \sigma^{3\infty}(a)$$

$$\widetilde{\sigma}^{3\infty}(a) = \cdots bacabaabacba \cdot$$
$$abacababacaba \cdots = \widetilde{\sigma}^{3\infty}(a) \quad : w'_a$$
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$$\widetilde{\sigma}^{3\infty}(a) = \cdots bacabaabacba\cdot \left\langle\begin{array}{ll} abacababacaba\cdots = \widetilde{\sigma}^{3\infty}(a) & : w'_a \\ bacabaabacaba\cdots = \widetilde{\sigma}^{3\infty}(b) & : w'_b \\ cabaabacababa\cdots = \widetilde{\sigma}^{3\infty}(c) & : w'_c \end{array}\right.$$

Properties [Queffelec, Holton-Zamboni 01]

- A primitive σ has finitely many singular bi-infinite words (left-special or right-special).
- Each singular word $w \in X_\sigma$ has an eventually periodic desubstitution path γ.

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Theorem [Gaborieau-Jaeger-Levitt-Lustig 98]

Let σ be a substitution and an iwip free group automorphism. Then

$$\mathrm{ind}(X_\sigma) \leq 2|A| - 2.$$

The substitution σ is parageometric if equality holds.

Example: Tribonacci is parageometric since $\mathrm{ind}(X_\sigma) = 4 = 2 \cdot 3 - 2$.

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Example: Tribonacci is parageometric since $\mathrm{ind}(X_\sigma) = 4 = 2 \cdot 3 - 2$.

Dictionary:

- σ free group automorphism $\to$ σ invertible

$$\sigma : a \mapsto ab, b \mapsto ac, c \mapsto a, \quad \sigma^{-1} : a \mapsto c, b \mapsto c^{-1}a, c \mapsto c^{-1}b$$

- σ irreducible with irreducible powers (iwip) if its powers fix no proper free factor

$$\sigma(a) = bac,\ \sigma(b) = ba,\ \sigma(c) = ca \quad \Rightarrow \quad \sigma(bc^{-1}) = bc^{-1}$$

Trees

Let $\sim$ be the transitive closure of the share-a-half relation

$$\Omega_\sigma = X_\sigma / \sim$$

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If σ is iwip and parageometric then Ω_σ is a tree. It is contained in the repelling tree $T_{\sigma^{-1}}$, an $\mathbb{R}$-tree which comes with a minimal action of F_A by isometries and a contracting homothety H.

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Map Q

There exists a unique continuous map $Q : X_\sigma \to \Omega_\sigma$ such that

$$Q(S(Z)) = Z_0^{-1} Q(Z),$$
$$Q(\sigma(Z)) = H(Q(Z)).$$

Singular desubstitution paths

For Tribonacci

w_a, w_b, w_c: $\begin{tikzcd} a \arrow[loop right, "\epsilon"] \end{tikzcd}$

w'_a: $\begin{tikzcd} a & a \arrow[l, "\epsilon"'] \arrow[rr, bend right, "a"'] & a \arrow[l, "\epsilon"'] & b \arrow[l, "\epsilon"'] \end{tikzcd}$

w'_b: $\begin{tikzcd} b & a \arrow[l, "a"'] & a \arrow[l, "\epsilon"'] \arrow[rr, bend right, "a"'] & a \arrow[l, "\epsilon"'] & b \arrow[l, "\epsilon"'] \end{tikzcd}$

w'_c: $\begin{tikzcd} c & b \arrow[l, "a"'] & a \arrow[l, "a"'] & a \arrow[l, "\epsilon"'] \arrow[rr, bend right, "a"'] & a \arrow[l, "\epsilon"'] & b \arrow[l, "\epsilon"'] \end{tikzcd}$

$$\varphi(w_a) = \varphi(w_b) = \varphi(w_c) = \mathbf{0}$$

$$\varphi(w'_a) = \varphi(w'_b) = \varphi(w'_c) = P$$

Tree substitutions

- Prototrees: $(W_a)_{a \in A}$ collection of simplicial trees, each with a finite set of gluing vertices V_a, $\forall\, a \in A$.

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- Initial patch: $W = \bigsqcup W_a / \sim$ where some gluing points are identified.
- $\tau(W_a)$ union of copies of some prototrees with some gluing points identified.
- τ maps gluing points of W_a to gluing points of $\tau(W_a)$, $\forall\, a$.

Tree substitutions

In our substitutive setting:

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$$\tau(W_a) = \bigsqcup_{a \xleftarrow{p,s} b} W_b / \sim$$

- The tree substitution τ maps a gluing point $P \in V_a$ with desubstitution path γ to

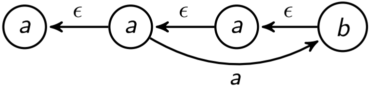
$$\tau(P) = P' + \pi(M_\sigma^{-1}(\ell(p))) \in V_b$$

where P' has desubstitution path $B(\gamma) =$ beheaded γ.

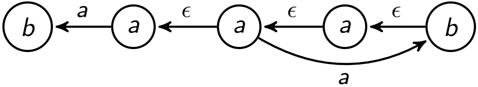
Singular points

Tribonacci substitution

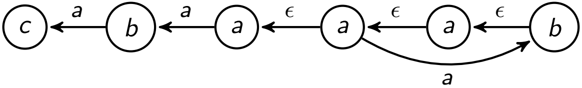
w'_a:



w'_b:



w'_c:



$$P = \varphi(w'_a) = \varphi(w'_b) = \varphi(w'_c)$$

Singular points

Tribonacci substitution

w'_a:
$$\begin{tikzcd} a & a \arrow[l, "\epsilon"'] \arrow[rr, bend right, "a"'] & a \arrow[l, "\epsilon"'] & b \arrow[l, "\epsilon"'] \end{tikzcd}$$

w'_b:
$$\begin{tikzcd} b & a \arrow[l, "a"'] & a \arrow[l, "\epsilon"'] \arrow[rr, bend right, "a"'] & a \arrow[l, "\epsilon"'] & b \arrow[l, "\epsilon"'] \end{tikzcd}$$

w'_c:
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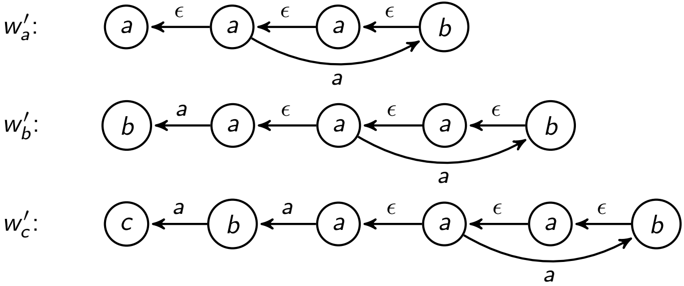
$$P = \varphi(w'_a) = \varphi(w'_b) = \varphi(w'_c)$$

Singular points

$$V_a = \{P, P + \pi(e_3), P + \pi(e_2)\}, \quad V_b = \{P, P + \pi(e_1)\}, \quad V_c = \{P\}$$

Singular points

Tribonacci substitution



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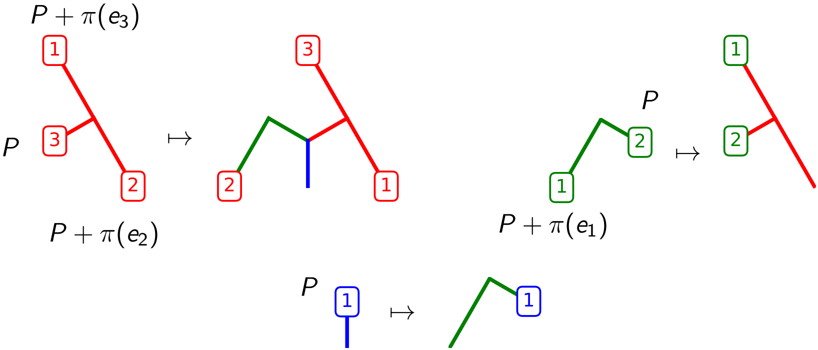
Gluing instructions

E.g. $P \mapsto P + \pi(e_3) \mapsto P + \pi(e_2) \mapsto P + \pi(e_1) \mapsto P + 2\pi(e_3) \mapsto \dots$

Tribonacci tree substitution

$$\boxed{\tau(W_a) = \bigcup_{a \xleftarrow{p,s} b} W_b + \pi(M_\sigma^{-1}\ell(p))}$$



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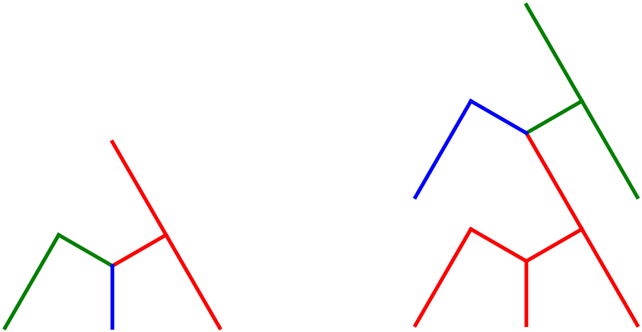
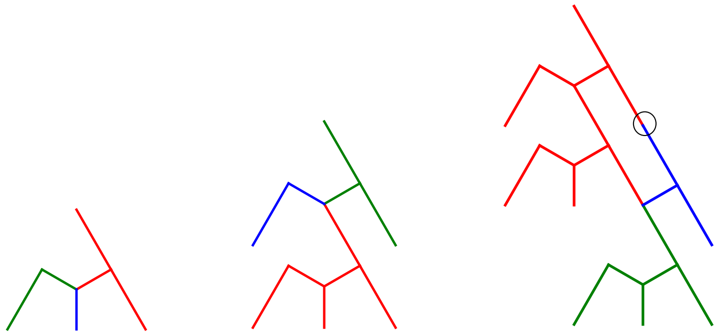


Loops

But. . .

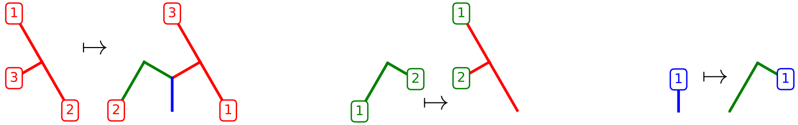


Tree substitution inside the Rauzy fractal

Two options to avoid loops:

- In the tree substitution some branches are dead-end: they remain leaves of the tree at each step. Then we prune those branches.
- Displace the singular points and use the addresses (desubstitution paths) to connect different cells.

Pruning and covering



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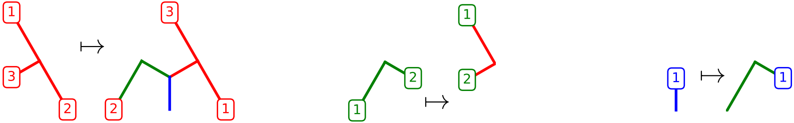

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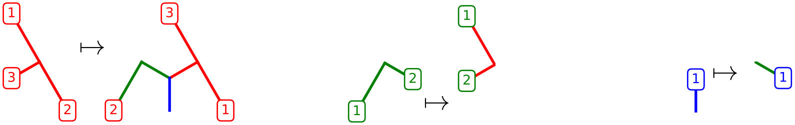

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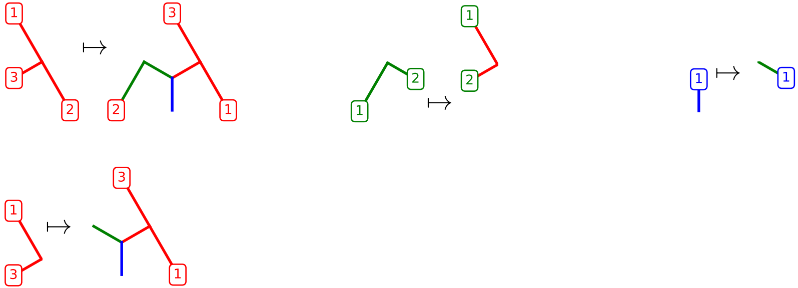

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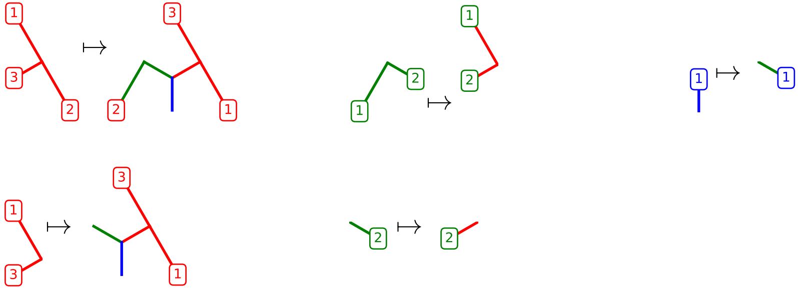

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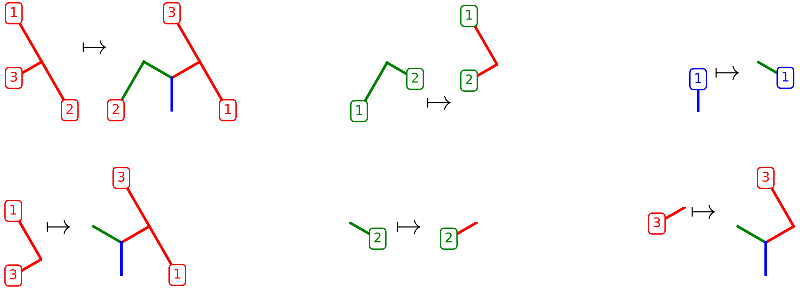


Pruned tree substitution

New tree substitution defined on more pieces: we have a new bigger alphabet A' with a forgetful map $f : A' \to A$, and for each $a \in A'$ the prototile W_a is a subtile of the original prototile $W_{f(a)}$.

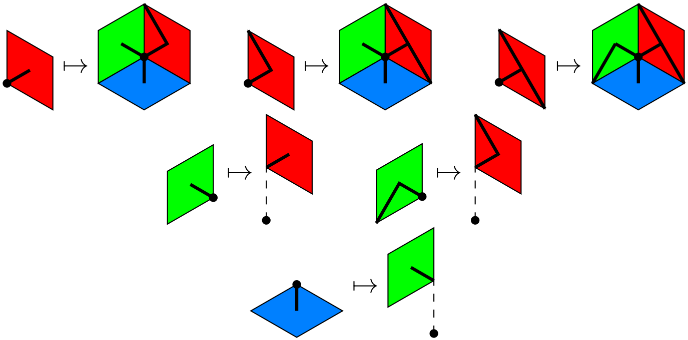
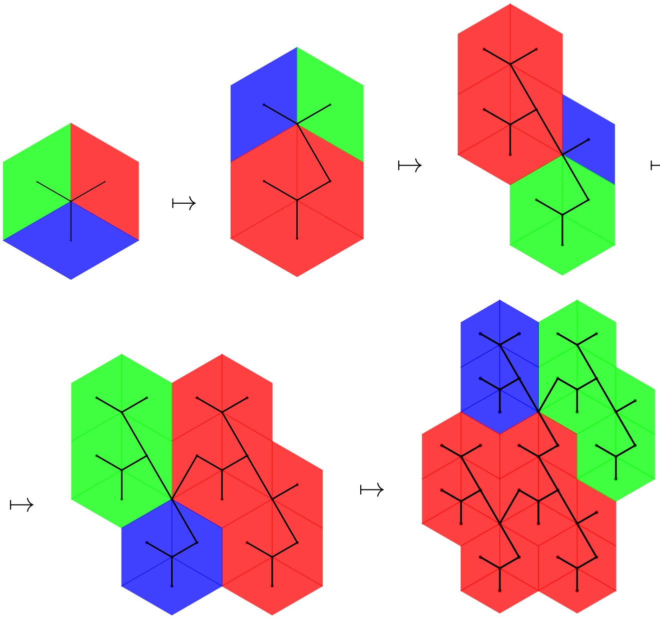


Figure: Pruned Tribonacci tree substitution plotted inside the dual faces

Generating the embedded tree



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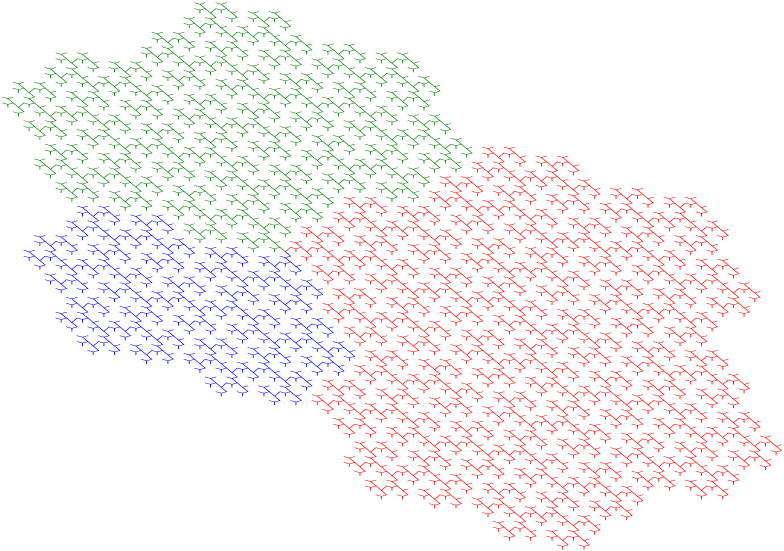
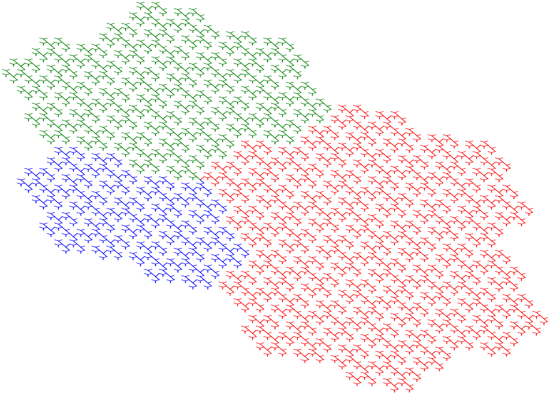


Embedding the tree



Conjecture

Up to covering the initial tree substitution, it is always possible to embed the tree inside the Rauzy fractal.

Motivation

(X_σ, S) substitutive subshift

Goal

Find an interval exchange transformation $([0,1), f)$ having (X_σ, S) as measure-theoretical factor:

$$
\begin{tikzcd}
{[0,1)} \arrow[r, "f"] \arrow[d] & {[0,1)} \arrow[d] \\
X_\sigma \arrow[r, "S"] & X_\sigma
\end{tikzcd}
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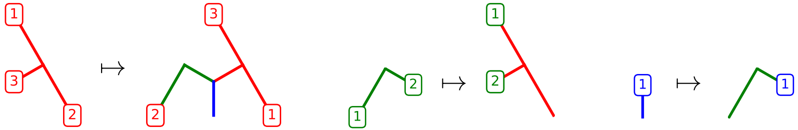
Examples:

- Arnoux-Yoccoz IET (Tribonacci dilation factor)
- Arnoux-Rauzy words (6-IETs)
- Bressaud-Jullian 2012

Contour of the tree

The abstract tree substitution



defines "naturally" the contour substitution on $\tilde{A}$

$$
\begin{aligned}
\chi : a_{12} &\mapsto a_{23} c_{11} b_{21} \\
a_{23} &\mapsto b_{12} a_{31} \\
a_{31} &\mapsto a_{12} \\
b_{12} &\mapsto a_{12} a_{23} \\
b_{21} &\mapsto a_{31} \\
c_{11} &\mapsto b_{21} b_{12}
\end{aligned}
$$

Characteristic polynomial of χ: $(x^3 - x^2 - x - 1)(x^3 + x^2 + x - 1)$.

Dual contour substitution

Tribonacci dual contour substitution

$$\begin{aligned}\chi^* : \quad a_{12} &\mapsto a_{31} b_{12}\\ a_{23} &\mapsto a_{12} b_{12}\\ a_{31} &\mapsto a_{23} b_{21}\\ b_{12} &\mapsto a_{23} c_{11}\\ b_{21} &\mapsto a_{12} c_{11}\\ c_{11} &\mapsto a_{12}\end{aligned}$$

χ^* is a covering of σ.

Vershik map

The **Vershik map**

$$V : \mathcal{P} \setminus \mathcal{P}_{\max} \to \mathcal{P} \setminus \mathcal{P}_{\min}$$

sends a prefix-suffix expansion

$$\gamma = a_0 \xleftarrow{p_0,\epsilon} a_1 \xleftarrow{p_1,\epsilon} \cdots a_r \xleftarrow{p_r,s_r} a_{r+1} \cdots a_{n-1} \xleftarrow{p_{n-1},s_{n-1}} a_n \cdots$$

where r is the smallest index such that $s_r = b_r t_r \neq \epsilon$, with $b_r \in A$ and $t_r \in A^*$, to

$$V(\gamma) = b_0 \xleftarrow{\epsilon,t_0} b_1 \xleftarrow{\epsilon,t_1} \cdots b_r \xleftarrow{p_r a_r,t_r} a_{r+1} \cdots a_{n-1} \xleftarrow{p_{n-1},s_{n-1}} a_n \cdots ,$$

with $\sigma(a_{r+1}) = p_r a_r s_r = p_r a_r b_r t_r$ and, for $i = 1, \ldots, r-1$, $\sigma(b_{i+1}) = b_i t_i$.

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Proposition

The Vershik map on $\mathcal{P}(\chi^*)$ is pushed forward by $Q_{\mathbb{S}^1}$ to a piecewise rotation of the unit circle $\mathbb{S}^1$.

All but finitely many adjacent arcs are rotated by the same angle.

Interval exchange on the circle

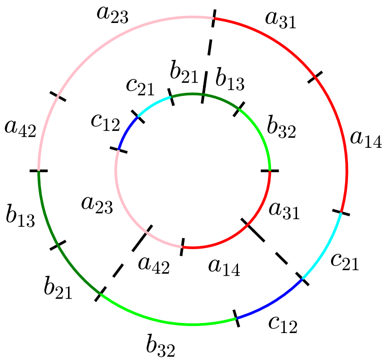

Figure: Inner circle = contour of W; outer = after piecewise exchange.

This is the Arnoux-Yoccoz interval exchange on the six intervals $a_{31}a_{14}$, $a_{42}a_{23}$, c_{12}, c_{21}, $b_{21}b_{13}$ and b_{32}.

Summary - Results

Let σ be a primitive substitution on a finite alphabet A and a parageometric iwip automorphism of the free group F_A.

Theorem 1 (Coulbois-M. 17)

There exists a tree substitution τ such that the iterations of τ on the initial tree W renormalized by the ratio of the contracting homothety converge to the repelling tree of the automorphism σ.
We provide an algorithm to construct this tree substitution.

Theorem 2 (Coulbois-M. 17)

If σ is irreducible Pisot, then the tree substitution τ can be realized inside the contracting space E_c of σ and the renormalized iterated images $M_\sigma^n \tau^n(W)$ converge to the Rauzy fractal $\mathcal{R}_\sigma$.
In all our examples we provide a covering tree substitution which yields trees inside the contracting space.

Summary - Results

Theorem 3 (Coulbois-M. 17)

There exist a contour substitution χ and a dual contour substitution χ^* such that:

- the substitutions χ and χ^* are primitive and the iterated images of the contour of W under χ converge to the contour of the compact heart Ω_A of the repelling tree;
- there exists a continuous map $Q_{\mathbb{S}^1} : X_{\chi^*} \to \mathbb{S}^1$ which pushes forward the action of the shift on X_{χ^*} to an interval exchange on the contour circle of Ω_A.

We provide an algorithm to compute χ and χ^*.

Questions and perspectives

- Parageometric $\Rightarrow$ Rauzy fractal arcwise connected. Disk-like?
- Non parageometric case.
- S-adic framework.
- Embedding of the tree in the contracting space.
- Spectra of the IET and of the substitutive subshift.

Thank you!