

Combinatorics of generalized exponents

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I. Generalized exponents in type A

$$\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i.$$

Let Q be the sublattice of $\mathbb{Z}^n$ generated by the vectors $e_i - e_{i+1}$, $1 \leq i < n$.

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, set $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$.

The q -Kostant partition function for $GL_n(\mathbb{C})$ is defined by

$$\prod_{1 \leq i < j \leq n} \frac{1}{1 - q^{\frac{x_i}{x_j}}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q(\beta) x^\beta.$$

- $\mathcal{P}_q(\beta) \in \mathbb{Z}_{\geq 0}[q]$

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- $\mathcal{P}_1(\beta)$ gives the number of nonnegative decompositions of β as a sum of $e_i - e_j, 1 \leq i < j \leq n$.
- $\mathcal{P}_q(\beta) = 0$ when $\beta \notin Q$.

A partition is a sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0) \in \mathbb{Z}^n$.

Each partition is encoded by its **Young diagram**. For example

$\lambda = (4, 3, 2, 1) \leftrightarrow$  and $|\lambda| = 10$.

Let $\rho = (n, n-1, \dots, 2, 1)$.

Let $\mathfrak{S}_n$ be the symmetric group of rank n .

The group $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by permutation $\sigma \cdot (\beta_1, \dots, \beta_n) = (\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)})$.

Definition

The generalized exponent $K_\lambda(q)$ is the polynomial of $\mathbb{Z}[q]$ s.t.

$$K_\lambda(q) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \mathcal{P}_q(\sigma(\lambda + \rho) - \rho - \mu)$$

where $\mu = \left(\frac{|\lambda|}{n}, \dots, \frac{|\lambda|}{n} \right)$.

A **semistandard tableau** T of shape λ is a filling of λ by letters in $\{1, \dots, n\}$ with

- strictly increasing columns from top to bottom
- weakly increasing rows from left to right.

Its **weight** is $\text{wt}(T) = (\mu_1, \dots, \mu_n)$ with $\mu_i = \# \text{ letters } i \text{ in } T$

Example

$T =$

1	1	2	4
2	3	5	
3	4		
5			

with weight $\text{wt}(T) = (2, 2, 2, 2, 2)$ and reading

$$w(T) = 4211532435$$

Theorem

- 1 $K_\lambda(1)$ is equal to the number of homogeneous SST of shape λ , i.e. with weight $\mu = \left(\frac{|\lambda|}{n}, \dots, \frac{|\lambda|}{n}\right)$.
- 2 $K_\lambda(q) \in \mathbb{Z}_{\geq 0}[q]$ and $K_\lambda(q) \neq 0$ if and only if n divides $|\lambda|$.

For Assertion 2 :

- sophisticated geometric or algebraic proofs (affine Kazhdan-Lusztig polynomials, Brylinsky-Kostant filtration).

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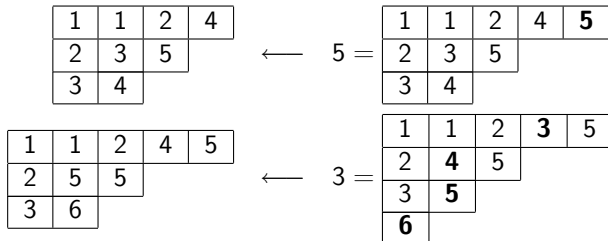
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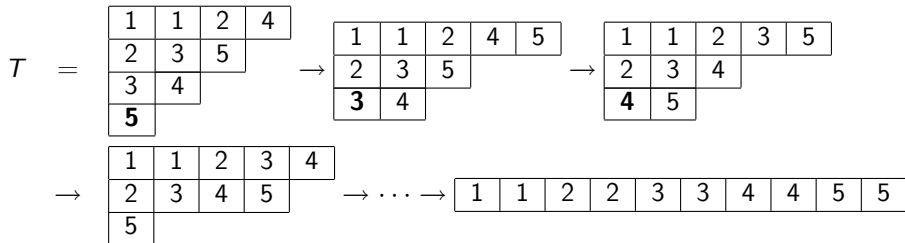
- sophisticated geometric or algebraic proofs (affine Kazhdan-Lusztig polynomials, Brylinsky-Kostant filtration).
- a combinatorial proof and description by Lascoux & Schützenberger...
- ...which extends in fact to any partition μ .

II. The charge statistics

Examples of row insertions



Cocyclage from T of weight μ :



Theorem

Cocyclage operations eventually ends to the unique row R_μ of weight μ

Definition

Set $\text{ch}_n(T) = \frac{(n-1)|\lambda|}{2} - l = \text{ch}_n(R_\mu) - l \geq 0$ where l the number of cocyclage operations needed to get R_μ .

Let $w = x_1 \cdots x_\ell$ be a word on $\{1 < \cdots < n\}$.

For each $i = 1, \dots, n-1$, form w_i the subword of w contained only the letters i and $i+1$.

$w_i^{\text{red}} = (i+1)^{\varepsilon_i(w)} i^{\varphi_i(w)}$ is obtained by recursive deletion of factors $i(i+1)$ in w_i .

Example: $w = 2421153243135$ with $n = 5$

- ① $w_1 = 21(12)1$ and $w_1^{\text{red}} = 211$. Thus $\varepsilon_1(w)$ and $\varphi_1(w) = 2$
- ② $w_2 = 2(23)(23)3$ and $w_2^{\text{red}} = \emptyset$. Thus $\varepsilon_2(w) = \varphi_3(w) = 0$.
- ③ $w_3 = 4(34)3$ and $w_3^{\text{red}} = 43$. Thus $\varepsilon_3(w) = \varphi_3(w) = 1$.
- ④ $w_4 = (45)(45)$ and $w_4^{\text{red}} = \emptyset$. Thus $\varepsilon_4(w) = \varphi_4(w) = 0$.

Theorem (LLT 1995)

When T is *homogeneous*, we have

$$\text{ch}_n(T) = \sum_{i=1}^{n-1} (n-i) \varepsilon_i(w(T)).$$

Theorem (LS 1980)

We have

$$K_{\lambda}(q) = \sum_{T \in SST_{\text{hom}}(\lambda)} q^{\text{ch}_n(T)}.$$

Example

For $\lambda = (2, 1, 0)$ we have

$$\text{ch}_n \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \right) = \text{ch}_n(213) = (3-1) \times 1 + (3-2) \times 0 = 2$$

$$\text{ch}_n \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \right) = \text{ch}_n(312) = (3-1) \times 0 + (3-2) \times 1 = 1$$

thus

$$K_{(2,1,0)}(q) = q + q^2.$$

Generalized exponents in type C and beyond

Start from

$$\prod_{1 \leq i \leq j \leq n} \frac{1}{1 - q^{x_i x_j}} \prod_{1 \leq i < j \leq n} \frac{1}{1 - q^{\frac{x_i}{x_j}}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{C_n}(\beta) x^\beta.$$

and replace $\mathfrak{S}_n$ by the group W_n of **signed permutations on** $\{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}$

$$w(x) = y \iff w(\bar{x}) = \bar{y}.$$

It acts on $\mathbb{Z}^n$ by

$$w \cdot (\beta_1, \dots, \beta_n) = (\beta'_1, \dots, \beta'_n) \text{ with } \beta'_i = \begin{cases} \beta_{w(i)} & \text{if } w(i) > 0 \\ -\beta_{-w(i)}, & \text{otherwise} \end{cases}$$

Definition

$$K_\lambda^{C_n}(q) = \sum_{w \in W_n} \varepsilon^{C_n}(w) \mathcal{P}_q^{C_n}(w(\lambda + \rho) - \rho).$$

In fact $K_\lambda^{C_n}(1)$ gives the **dimension of the zero weight space** in the irreducible $SP_{2n}(\mathbb{C})$ -representation indexed by λ . Thus :

- $K_\lambda^{C_n}(1) = \#$ **King tableaux** of shape λ and zero weight,
- $K_\lambda^{C_n}(1) = \#$ **Kashiwara-Nakashima tableaux** of shape λ and zero weight
- $K_\lambda^{C_n}(1) = \#$ **Littelmann paths** of shape λ starting and ending on 0.
- This generalizes to any weight.

Problem

Find a charge for type C_n which proves the positivity of the coefficients.

From [Cauchy and Littlewood identities](#) on symmetric functions one gets:

Theorem (C.L., C. Lenart)

1

$$\frac{K_{\lambda}(q)}{\prod_{i=1}^n (1 - q^i)} = \sum_{\gamma \in \mathcal{P}_n} q^{|\gamma|} c_{\gamma, \gamma^*}^{\lambda}.$$

Here

- $\gamma^* = (0, \gamma_1 - \gamma_2, \dots, \gamma_1 - \gamma_n)$,
- the $c_{\gamma, \gamma^*}^{\lambda}$'s are the [Littlewood-Richardson coefficients](#),
- the c_{ν}^{λ} 's are the [branching coefficients](#) appearing in the restrictions
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2

$$\frac{K_{\lambda}^{C_n}(q)}{\prod_{i=1}^n (1 - q^{2i})} = \sum_{\nu \in 2\mathcal{P}_{2n}} q^{|\nu|/2} c_{\nu}^{\lambda}$$

Here

- $\gamma^* = (0, \gamma_1 - \gamma_2, \dots, \gamma_1 - \gamma_n)$,
- the $c_{\gamma, \gamma^*}^{\lambda}$'s are the [Littlewood-Richardson coefficients](#),
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 $V(\nu) \downarrow_{SP_{2n}}^{GL_{2N}}$

- The combinatorial description of $K_\lambda(q)$ (for GL_n) with the ε_i 's easily follows from 1.

Example

$$K_{\square}^{C_\infty}(q) = \sum_{k \geq 1} q^{2k} = \frac{q^2}{1 - q^2}.$$

- The combinatorial description of $K_\lambda(q)$ (for GL_n) with the ε_i 's easily follows from 1.
- **Very more difficult** for SP_{2n} because c_ν^λ has a simple description only when $\nu_k = 0$ for $k > n$ where

$$c_\nu^\lambda = \sum_{\delta \in \mathcal{P}_n^\square} c_{\lambda, \delta}^\nu.$$

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- Only gets a description of **the formal series** $K_\lambda^{C_\infty}(q) = \lim_{n \rightarrow +\infty} K_\lambda^{C_n}(q)$.

Example

$$K_{\square}^{C_\infty}(q) = \sum_{k \geq 1} q^{2k} = \frac{q^2}{1 - q^2}.$$

A “King tableau” T of shape $\lambda = (\lambda_1, \dots, \lambda_n)$ is a filling of λ by letters of

$$\left\{ 1 < \frac{2}{\overline{1}} < 3 < \frac{4}{\overline{2}} < \dots < 2n-1 < \frac{2n}{\overline{n}} \right\}$$

s.t.

- T is semistandard,
- The letters in row i are greater to $2i-1$

Such a tableau is **distinguished** when we have moreover

- $\varphi_i(T) = 0$ for any odd i

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Such a tableau is **distinguished** when we have moreover

- $\varphi_i(T) = 0$ for any odd i
- $\varepsilon_i(T)$ is even for any odd i .

Theorem (C. L. and C. Lenart)

We have

$$K_{\lambda}^{C_n}(q) = \sum_{T \text{ distinguished of shape } \lambda} q^{\text{ch}^{C_n}(T)}$$

where

$$\text{ch}^{C_n}(T) = \sum_{i=1}^{2n-1} (2n-i) \left\lfloor \frac{\varepsilon_i(T)}{2} \right\rfloor.$$

Corollary

- ① $K_{\lambda}^{C_n+1}(q) - K_{\lambda}^{C_n}(q)$ has *nonnegative coefficients*.
- ② Determination of the *highest and lowest monomials* in $K_{\lambda}^{C_n}$.
- ③ Simple formulas for particular partitions λ .

Futures directions

- ① Other classical types for finite n .
- ② Generalization to any dominant weight $\mu \neq 0$.
- ③ Connect to a conjectural charge statistics defined on KN tableaux from cyclage operation.
- ④ Parabolic cases (positivity not even known in general).