Mélange et taux de croissance des pavages

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with Silvère Gangloff

Journées Combinatoires de Bordeaux

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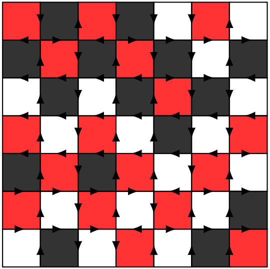
$\#L_n$: number of square patterns of side length n

- Square ice model

$$\#\mathcal{L}_n \approx (4/3)^{3n^2/2}$$

(Lieb 1967)

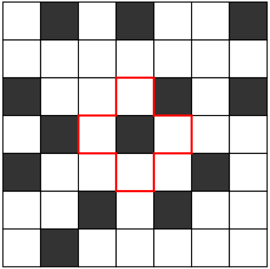


$\#L_n$: number of square patterns of side length n

- Square ice model
- Hard square model

$$\#\mathcal{L}_n \approx (1.5032 \pm 0.0003)^{n^2}$$

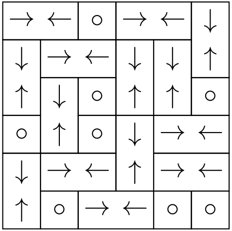
(Calkin, Wilf 1997)



Counting patterns

$\#L_n$: number of square patterns of side length n

- Square ice model
- Hard square model
- Monomer-dimer cover



$$\#\mathcal{L}_n \approx (1.9402\ldots)^{n^2}$$

(Baxter 1968)

Dimer only :

$$\#\mathcal{L}_n \approx e^{Gn^2/\pi}$$

(Kasteleyn, Fischer, Temperley 1961)

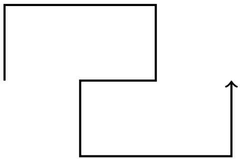
Counting patterns

$\#L_n$: number of square patterns of side length n

- Square ice model
- Hard square model
- Monomer-dimer cover
- Self-avoiding walks

$$\#\mathcal{L}_n \approx (2.65 \pm 0.03)^n$$

(Pönitz, Tittman 2000)



$\downarrow$

N	O	O	S	E	S	O	O	N

$\mathcal{A}$ an finite **alphabet** ;

$\mathcal{A}^*$ the set of (d-dimensional) finite **patterns** (**words** if $d = 1$).

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A **tiling space** Σ is defined by a set of **forbidden patterns** $\mathcal{F}$. A pattern is **admissible** for Σ if it contains no forbidden pattern.

Σ is of **finite type** if $\mathcal{F}$ is finite.

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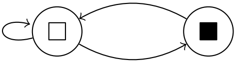
Combinatorial entropy

$\mathcal{L}_n(\Sigma)$: admissible patterns on an hypercube of side length n.

$$\frac{\log \#\mathcal{L}_n(\Sigma)}{n^d} \searrow h(\Sigma).$$

$\Leftrightarrow$ Exponential growth rate of admissible patterns.

$\mathcal{F}$	$\#\mathcal{L}_n$	h
$\{■■\}$	$\sim \varphi^n$	$\log \varphi$



Admissible words $\leftrightarrow$ Paths in a graph

 $\in \Sigma$

 $\notin \Sigma$

$\mathcal{F}$	$\#\mathcal{L}_n$	h
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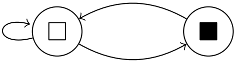


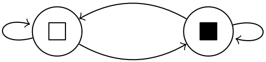
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \text{dominant eigenvalue : } \varphi \longrightarrow h(\Sigma) = \log \varphi$$

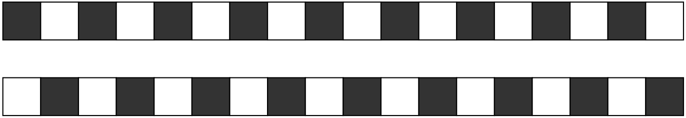
 $\in \Sigma$

 $\notin \Sigma$

Examples

$\mathcal{F}$	$\#\mathcal{L}_n$	h
$\{\blacksquare\blacksquare\}$	$\sim \varphi^n$	$\log \varphi$
$\{\square\square, \blacksquare\blacksquare\}$	2	0





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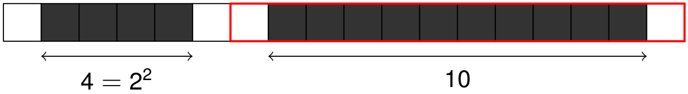
$\{\square\blacksquare^n\square : n \notin \{k^2 : k \in \mathbb{N}\}\}$	?	$0,774 \pm 0,03$



Entropy ?

Thermodynamical entropy	Discrete models is statistical physics ; $\rightarrow$ Crystallography, Ising model...
Information entropy	Average information content in bits ; $\rightarrow$ storage capacity, maximal compression rate.
Topological entropy	Invariant in dynamical systems theory. $\rightarrow$ John Milnor, 2002 : *"Given an explicit dynamical system and* $\varepsilon > 0$*, is it possible to compute the associated entropy [...] with a maximum error of* ε *?"*

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Questions

Consider a family of tiling spaces.

- How hard is computing the entropy ?
- What values can the entropy take ?

Computing the entropy

Σ a tiling space defined by $\mathcal{F}$.

$$\frac{\log \#\mathcal{L}_n(\Sigma)}{n^d} \searrow h(\Sigma).$$

How good of an approximation is $n \mapsto \frac{\log \#\mathcal{L}_n(\Sigma)}{n^d}$?

Real computability

A real number α is :

- **computable** if there is an algorithm $n \mapsto \alpha_n$ such that

$$|\alpha - \alpha_n| \leq 2^{-n};$$

- **upper-semi-computable** if instead

$$\alpha_n \searrow \alpha.$$

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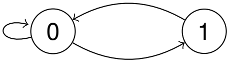
$\mathcal{F} \mapsto h(\Sigma)$ is **upper-semi-computable**.

Finite type, dimension 1

The entropy of **one-dimensional finite type** tiling spaces is **computable**.



$$\longrightarrow \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \longrightarrow \quad h(\Sigma) = \log \varphi$$

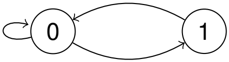
(even a nice close formula)

Finite type, dimension 1

The entropy of **one-dimensional finite type** tiling spaces is **computable**.

Lind 1974

Any real number $q \log p$ with $q \in \mathbb{Q}$ and p a Perron number can be realised as the entropy of such a system.



$$\longleftarrow \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \longleftarrow \quad h(\Sigma) = \log \varphi$$

(even a nice close formula)

Hochman, Meyerovitch 2007

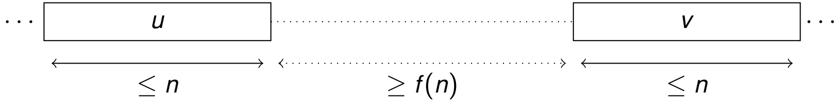
Any **upper-semi-computable** real number can be realised as the entropy of a **two-dimensional finite type** tiling space.

In particular, the entropy of these systems is uncomputable in general.

- Computational proof by simulating Turing machines ;
- Doesn't say anything about specific models.

Mixing properties

Σ is f**-irreducible** if two admissible patterns of diameter $\leq n$ can be “glued” at any distance $\geq f(n)$ into an admissible pattern.



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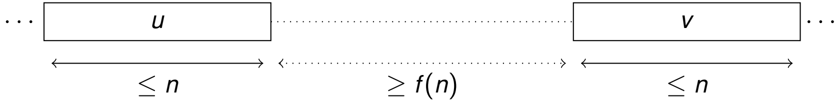
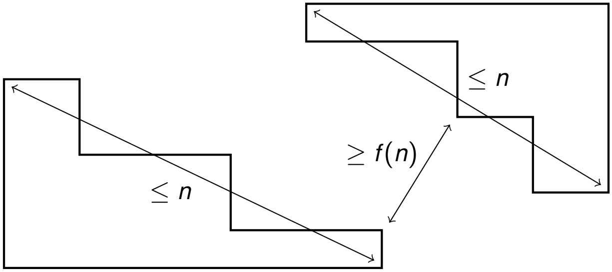
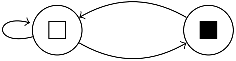




Examples

$\mathcal{F}$	$\#\mathcal{L}_n$	h	rate
$\{$■■$\}$	$\sim \varphi^n$	$\log \varphi$	1



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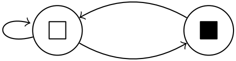

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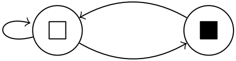

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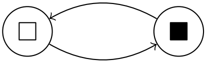

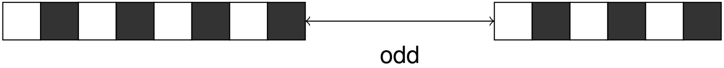

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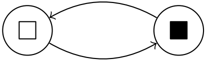

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$n^2 + 1$

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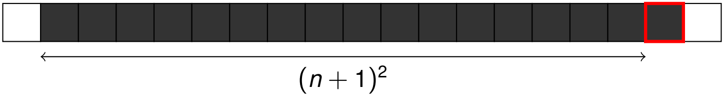

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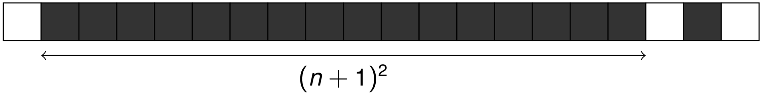
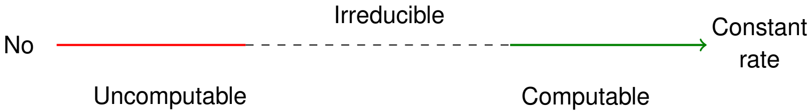


Finite type, higher dimension

Hochman, Meyerovitch 2007

The entropy of higher-dimensional finite type $O(1)$**-irreducible** tiling spaces is **computable**,
Pavlov et Schraudner 2015 : and even EXPTIME in dimension 2.



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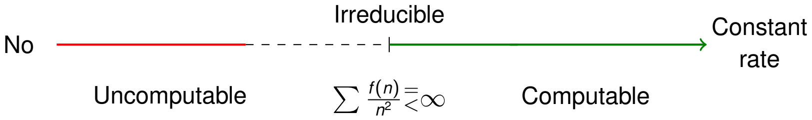
Gangloff, H.

The entropy of higher-dimensional finite type $O(f)$**-irreducible** tiling spaces, where $\sum_n \frac{f(n)}{n^2} < +\infty$, is **computable**.



General tiling spaces do not have a finite description.
We consider **effective** tiling spaces, i.e.

$$w \mapsto w \in \mathcal{F}?$$

is (algorithmically) decidable.

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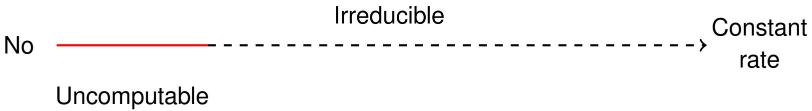
is (algorithmically) decidable.

Grillenberger 73 ; indep. Simonsen 06, Hertling et Spandl 07

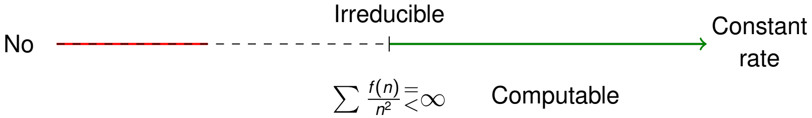
Any upper-semi-computable real number can be realised as the entropy of an **effective** tiling space in any dimension.

Therefore the entropy of these systems is **uncomputable**.



Gangloff, H.

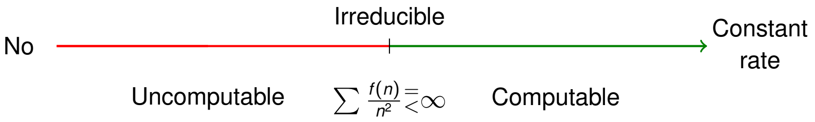
1. If $\sum \frac{f(n)}{n^2} < +\infty$, then the entropy of effective $O(f)$-**irreducible** tiling spaces is **computable**.



Gangloff, H.

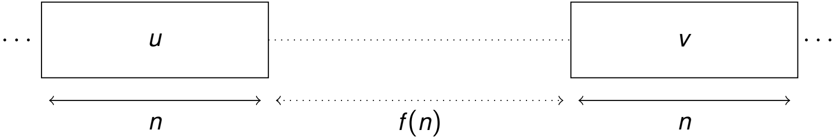
1. If $\sum \frac{f(n)}{n^2} < +\infty$, then the entropy of effective $O(f)$-**irreducible** tiling spaces is **computable**.

2. If $\sum \frac{f(n)}{n^2} = +\infty$, then any upper-semi-computable real number can be realised as the entropy of an effective $O(f)$-**irreducible** tiling space and the entropy of these systems is **uncomputable** in general.



Computing the entropy of irreducible tiling spaces

$$\frac{\log \#\mathcal{L}_n(\Sigma)}{n} \searrow h(\Sigma).$$



$$\#\mathcal{L}_n(\Sigma)^2 \leq \#\mathcal{L}_{2n+f(n)}(\Sigma) \leq \#\mathcal{L}_{2n}(\Sigma) \cdot 2^{f(n)}$$

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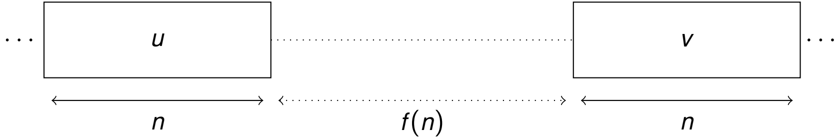

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$$\frac{\log \#\mathcal{L}_n(\Sigma)}{n} \leq \frac{\log \#\mathcal{L}_{2n}(\Sigma)}{2n} + \frac{f(n)}{2n}$$

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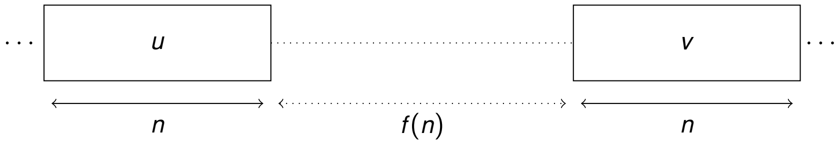

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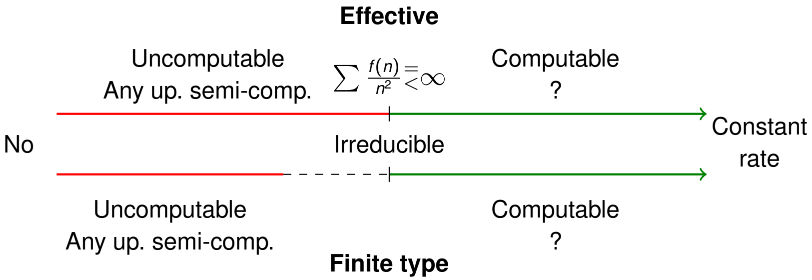
$$\frac{\log \#\mathcal{L}_{2^n}(\Sigma)}{2^n} \leq h(\Sigma) + \boxed{\sum_{k \geq n} \frac{f(2^k)}{2^{k+1}}}$$

Error term
converges iff $\frac{f(n)}{n^2} < \infty$

Conclusion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$\sum \frac{f(n)}{n^2} \substack{= \\ <} \infty$
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?
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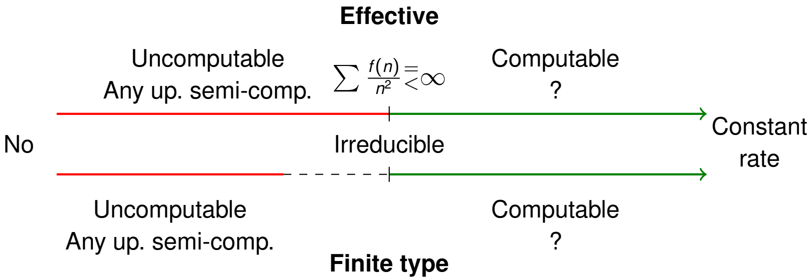
Finite type



Questions

- Finish the finite type picture ? (ask S. Gangloff)
- Posible entropies under the threshold ?

Approach

- From combinatorial to algorithmic to computational ;
- Fixing the tiling complexity to unveil the effect of mixing properties.