Méthodes galoisiennes pour l'indépendance différentielle de séries génératrices

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Journées Combinatoire de Bordeaux

Question : Decide if a function $f = \sum_{n\in\mathbb{N}} a_n x^n$ is not holonomic ?

Analytic strategy : Show that f does not fulfill some regularity criteria :
An holonomic function has

- *Finite number of singularities*
  Generating series for quarter plane walks : Kernel Method and action of the group of the Walk (for instance strategy of Bousquet-Mélou and Petkovsek)
- *Good asymptotic :* $a_n = O(n!^d)$ for some integer d
  Non holonomy of $P(x) := \sum_{n\in\mathbb{N}} p_n x^n$ with p_n is the integer partition

$$p_n \simeq \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3n}}.$$

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A function $f(x)$ is *differentially algebraic* over $\mathbb{C}(x)$ if there exist $n \in \mathbb{N}$ and $P \in \mathbb{C}(x)[X_0, \ldots, X_n]^*$ such that $P(f, f', \ldots, f^{(n)}) = 0$.

The class of Diff. alg. functions is too *wild* from an analytic point of view.

- *Singularities :*
  - $\zeta(x) = \sum \frac{1}{n^x}$ is diff. trans. and has one single pole.
  - $P(x) = \prod_{k \in \mathbb{N}^*} \frac{1}{1-x^k}$ is diff. alg and has infinite number of singularities.
- *Asymptotics :* $|a_n| \leq e^{(d+\epsilon) n log(n)}$ for all $\epsilon > 0$ (Sibuya-Sperber)

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The discrete equation

Many of the generating series satisfy *discrete equations*.
For instance :

- $\sum_{n\in\mathbb{N}} a_n x^n$ with (a_n) a sequence of p-automatic numbers satisfies a linear p-Mahler equation :

$$f(x^{p^n}) + a_{n-1}(x) f(x^{p^{n-1}}) + \cdots + a_0(x) = 0.$$

- $Q(x,y,z,t)$ the generating series, counting walks in the quarter plane, is related to a function $f(\omega)$ that satisfies
  - Genus zero :

$$f(q\omega) = f(\omega) + b(\omega)$$

    with $q \in \mathbb{R} \setminus \pm\{1\}$ and $b(\omega) \in \mathbb{C}(\omega)$.
  - Genus 1 :

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Algebraic strategy

Use the discrete equation
to determine
a potential set of algebraic or differentially algebraic relations
for the generating series.

Elementary Proof of Transcendence of l_q

Suppose to the contrary that $l_q(x)$ that satisfies $l_q(qx) = l_q(x) + 1$ is algebraic. Let

$$l_q(x)^n + a_{n-1}(x) l_q(x)^{n-1} + \ldots + a_0(x) = 0, \qquad (2.1)$$

be a minimal relation with $a_i(x) \in \mathbb{C}(x)$, $a_0(x) \neq 0$, $n > 0$.
Change x into qx in (2.1) to find

$$l_q(qx)^n + a_{n-1}(qx) l_q(qx)^{n-1} + \ldots + a_0(qx) = 0. \qquad (2.2)$$

Use $l_q(qx) = l_q(x) + 1$ in (2.2) and eliminate $l_q(x)^n$ between (2.1) and (2.2),we find

$$(a_{n-1}(qx) - a_{n-1}(x) + n) l_q(x)^{n-1} \ldots + (b_0(x)) = 0.$$

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Note that

$$a_{n-1}(qx) - a_{n-1}(x) = -n = -n * 1 = -n(l_q(qx) - l_q(x))$$

$$\Updownarrow$$

$$(l_q(qx) - \frac{a_{n-1}(qx)}{n}) = (l_q(x) - \frac{a_{n-1}(x)}{n})$$

That is $l_q(x) - \frac{a_{n-1}(x)}{n}$ is an elliptic function, a meromorphic function fixed by x goes to qx !

Galois theory of difference equations
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this naive elimination process
in order to determine

1. the potential algebraic relations for the series : $l_q(x) - \frac{a_{n-1}(x)}{n}$ is an elliptic function.
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$$l_q(qx) - l_q(x) = 1 \text{ and } l_q \text{ algebraic}$$

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$$l_q(qx) - l_q(x) = 1 \text{ and } l_q \text{ algebraic}$$

$$\Downarrow$$

$$-n.1 = a_{n-1}(qx) - a_{n-1}(x) \text{ with } a_{n-1}(x) \in \mathbb{C}(x).$$

Abstract setting : A σ-δ-field K is a field with an automorphism σ and a derivation δ commuting with σ.
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Let K be a σ-δ-field and let $b \in K$.
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Consider the walks in the quarter plane starting from $(0,0)$ with steps in a fixed set

$$\mathcal{D} \subset \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}.$$

Example with possible directions

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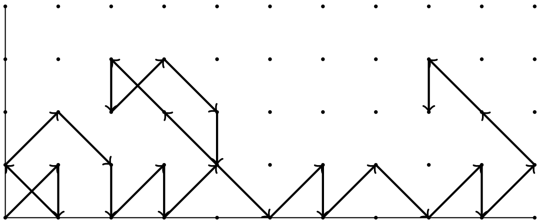

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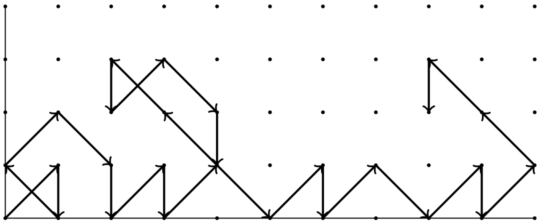

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Application to the walks

Let $\mathcal{W}$ be a walk with $\mathcal{D} \subset \{(i,j)\ i,j = -1,0,1\}$ and weights $d_{i,j} \in \mathbb{Q}$.
The generating series $Q_{\mathcal{W}}(x,y,t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \to^{k} (l,s)\right) x^{l} y^{s} t^{k}$ satisfies

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The *Kernel* $E_{\mathcal{W}}$ of the walk is

$$E_{\mathcal{W}} = \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 | K_{\mathcal{W}}(x, y, t) = 0\}.$$

Therefore, either

- $E_{\mathcal{W}}$ is reducible ;
- $E_{\mathcal{W}} \simeq \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ is an elliptic curve (genus 1) ;
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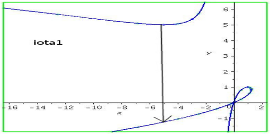
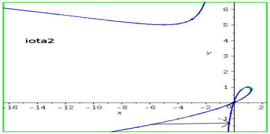
Group of the Walk

Fayolle, Iasnorodski, Malyshev (1999), Bousquet-Mélou, Mishna (2010) define and classify the group of the walk.

$$E_{\mathcal{W}} = \overline{\{(x,y) \mid K_{\mathcal{W}}(x,y,t)=0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$



$$\iota_1(x,y) = \left(x, \frac{1}{y}\frac{\sum_{(i,-1)\in\mathcal{D}} x^i}{\sum_{(i,+1)\in\mathcal{D}} x^i}\right)$$



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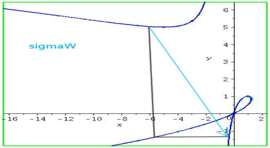

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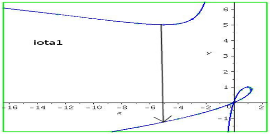

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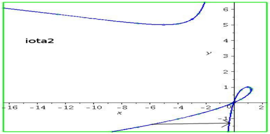

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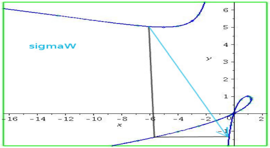

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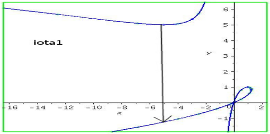

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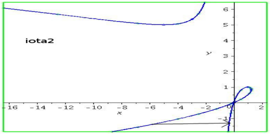

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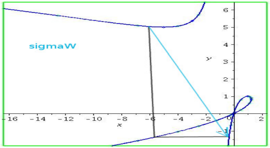

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Results : For the **79** unweighted walks

- $|G_{\mathcal{W}}| < \infty$ for **23** walks $\Rightarrow Q_{\mathcal{W}}(x, y, t)$ algebraic or holonomic.
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Caracterization of the genus zero case (D.-H.-R.-S.)

Fix $t \notin \mathbb{Q}$

Up to symmetries, all the weighted genus zero walks corresponds to the following set of directions

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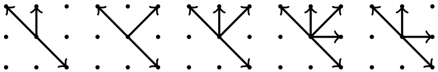

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Genus 0 (D.H.R.S.) Let $\phi : \mathbb{P}^1(\mathbb{C}) \to E_{\mathcal{W}}, \omega \mapsto (x(\omega), y(\omega))$ birational such that

- $\phi : \mathbb{P}^1 \setminus \phi^{-1}(\Omega) \to E_{\mathcal{W}} \setminus \{\Omega\}$ is a bijection and $\phi^{-1}(\Omega) = \{0, \infty\}$
- The following diagrams are commutative

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\phi"] \arrow[d, "\omega \mapsto \frac{1}{\omega}"] & E_{\mathcal{W}} \arrow[d, "\iota_1"] \\ \mathbb{P}^1 \arrow[r, "\phi"] & E_{\mathcal{W}} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}^1 \arrow[r, "\phi"] \arrow[d, "\omega \mapsto \frac{q}{\omega}"] & E_{\mathcal{W}} \arrow[d, "\iota_2"] \\ \mathbb{P}^1 \arrow[r, "\phi"] & E_{\mathcal{W}} \end{tikzcd}$$

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The dynamical functional equation-genus zero (D-H-R-S)

Fix $0 < t < 1$ transcendental over $\mathbb{Q}$ and set $F_{\mathcal{W}}(x,y,t) = K_{\mathcal{W}}(x,y,t)Q_{\mathcal{W}}(x,y,t)$

We have for $|x|, |y| < 1$,

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$$f(q\omega) - f(\omega) = b(\omega) \in \mathbb{C}(\omega)$$

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The dynamical functional equation-genus 1 (Kurkova-Rashel)

For the genus 1 walks, one finds a parametrization $\phi : \mathbb{C} \to E_{\mathcal{W}}, \omega \mapsto (x(\omega), y(\omega))$ and $\omega_1, \omega_2, \omega_3 \in \mathbb{C}^*$ such that

- $E_{\mathcal{W}} \simeq \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and ϕ is ω_1-ω_2-periodic and

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Let C_E be the field of ω_1-ω_2-periodic.

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From Galois theoretic criteria, it follows :
The generating series $Q(x, y, t)$ is differentially algebraic if

- Genus 1 : there exist g an ω_1-ω_2-periodic function, $c_0, \ldots, c_{n-1} \in \mathbb{C}$ not all zero and $n \geq 0$ such that

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Lemma (Chen-Singer 2012)

Let $q \in \mathbb{C}$, not a root of unity and $b(\omega) \in \mathbb{C}(\omega)$. The following statement are equivalent

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- *There exists $g \in \mathbb{C}(\omega)$ such that $b(\omega) = g(q\omega) - g(\omega)$*
- *The q-orbit residues of b at non zero poles are equal to zero and b has no constant terms in its partial fraction decomposition.*

Example for q-orbit residues : $b(\omega) = \frac{1}{\omega-1} + \frac{-1}{(\omega-1)^2} + \frac{2}{(q\omega-1)^2}$

- the poles $\{1, 1/q\}$ belong to the same q-orbit, i.e., $1/q \in q^{\mathbb{Z}}.1$
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Corollaire

If a rational function $b(\omega)$ *has*

1. *a pole* ω_0 *of order* m *;*
2. *such that there is no other pole of* $b(\omega)$ *in* $q^{\mathbb{Z}}\omega_0$ *of order greater than or equal to* m*,*

then there is no $g(\omega)$ *such that* $b(\omega) = g(q\omega) - g(\omega)$*.*

Prove that $Q(x, y, t)$ is diff. trans. :
Suppose to the contrary that $Q(x, y, t)$ is diff. alg. Then,

- $f(\omega)$ is diff. alg. (Parametrization of the Kernel)
- $b(\omega) = x(\iota_1(y) - y) \circ \phi(\omega)$ has a telescoper (Galoisian Criteria)
- If $x(\iota_1(y) - y)$ has a pole in $\mathbb{P}^1 \times \mathbb{P}^1$ of order m, isolated on its $\sigma_{\mathcal{W}}$-orbit then $b(\omega)$ has no telescoper. (Residue criteria for telescopers)

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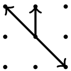
The following theorem holds

Theorem (D-H-R-S)

For all the genus zero walks, the generating series $Q(x, y, t)$ is differentially transcendental.

An example

Consider the unweighted walk  with

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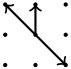
Consider the unweighted walk  with

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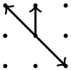
Consider the unweighted walk  with

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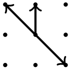
Consider the unweighted walk  with

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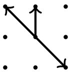
Consider the unweighted walk  with

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$$\phi \text{ is a bijection outside } ([0:1],[0:1])$$
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$$b(\omega) \text{ has no telescoper!}$$
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Generically $x(\iota_1(y) - y)$ *has*

- 6 simple poles $\{P_1, P_2, Q_1, Q_2, \sigma_{\mathcal{W}}(Q_1), \sigma_{\mathcal{W}}(Q_2)\}$
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Idea of proof :

- Find ω_1, ω_2 such that $P_1 = \phi(\omega_1)$ and $P_2 = \phi(\omega_2)$
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Theorem

For a genus 1 walk $\mathcal{W}$, the following are equivalent

- $Q(x,y,t)$ *is diff. alg ;*
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The orbit is w.r.t. the action of $\sigma_{\mathcal{W}}$ on the elliptic curve $E_{\mathcal{W}}$.

- $\sigma_{\mathcal{W}}(X) = X \oplus P$ for $P = \phi(\omega_3)$;
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- $X_1 = \sigma_{\mathcal{W}}^n(X_2) \Leftrightarrow X_1 = X_2 \oplus nP$ can be decided by computer.
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Theorem (D.-H.-R.-S.)

Except 9 exceptional cases, all the generating series for unweighted walks of genus 1 with infinite group are diff. trans.
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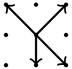
$\mathcal{W} =$  $E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : x_0 x_1 y_0 y_1 - t(x_0^2 y_1^2 + x_0 x_1 y_1^2 + x_1^2 y_0^2 + x_0^2 y_0^2) = 0$

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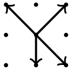
$\mathcal{W} =$  $\quad E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : x_0 x_1 y_0 y_1 - t(x_0^2 y_1^2 + x_0 x_1 y_1^2 + x_1^2 y_0^2 + x_0^2 y_0^2) = 0$

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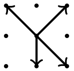
$\mathcal{W} =$  $\quad E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : x_0x_1y_0y_1 - t(x_0^2y_1^2 + x_0x_1y_1^2 + x_1^2y_0^2 + x_0^2y_0^2) = 0$

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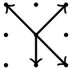
$\mathcal{W} =$  $\quad E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : x_0x_1y_0y_1 - t(x_0^2y_1^2 + x_0x_1y_1^2 + x_1^2y_0^2 + x_0^2y_0^2) = 0$

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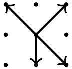
$\mathcal{W} =$  $E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : x_0 x_1 y_0 y_1 - t(x_0^2 y_1^2 + x_0 x_1 y_1^2 + x_1^2 y_0^2 + x_0^2 y_0^2) = 0$

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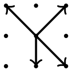
$\mathcal{W} =$  $E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : x_0x_1y_0y_1 - t(x_0^2y_1^2 + x_0x_1y_1^2 + x_1^2y_0^2 + x_0^2y_0^2) = 0$

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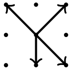
$\mathcal{W} =$  $E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : x_0x_1y_0y_1 - t(x_0^2y_1^2 + x_0x_1y_1^2 + x_1^2y_0^2 + x_0^2y_0^2) = 0$

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