The 0-Rook and 0-Renner Monoids

Joël Gay
joint work with Florent Hivert

Université Paris-Sud, LRI & École Polytechnique, LIX

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Contents

Symmetric Group

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Permutation	5 4 2 3 1	2 3 5 4 1

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$$\begin{array}{ccc}
\text{Permutation} & \begin{array}{ccccc} 5 & 4 & 2 & 3 & 1 \end{array} & \begin{array}{ccccc} 2 & 3 & 5 & 4 & 1 \end{array} \\
\text{Permutation matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\end{array}$$

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Can be seen as matrices.

$$\begin{array}{ccc} \text{Permutation} & \begin{array}{ccccc} 5 & 4 & 2 & 3 & 1 \end{array} & \begin{array}{ccccc} 2 & 3 & 5 & 4 & 1 \end{array} \\ \text{Permutation matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Product of $\mathfrak{S}_n$: product of matrices

Generating set

For $1 \leq i \leq n-1$, $s_i := (i, i+1)$

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$$3726145 \cdot s_5 = 3726415$$
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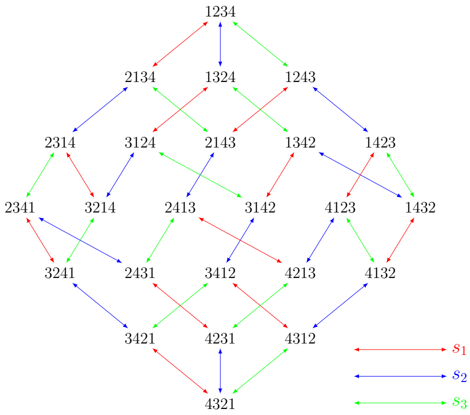
For $1 \leq i \leq n-1$, $s_i := (i, i+1)$

$$\mathfrak{S}_n = \langle s_1, \ldots s_{n-1} \rangle$$

$$3726\textcolor{red}{14}5 \cdot s_5 = 3726\textcolor{red}{41}5$$
$$3726\textcolor{red}{41}5 \cdot s_5 = 3726\textcolor{red}{14}5$$



Permutohedron

Hecke monoid at $q = 0$

$$\mathfrak{S}_n$$

Symmetric Group

Hecke monoid at $q = 0$

$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q)$$

Symmetric Group

Iwahori-Hecke algebra

Hecke monoid at $q = 0$

$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q) \xrightarrow{q=0} H_n^0$$

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Symmetric Group	Iwahori-Hecke algebra	Hecke monoid at $q = 0$
$\mathfrak{S}_n = \langle s_1, \ldots s_{n-1} \rangle$	$T_i = q s_i - (1 - q)\pi_i$	$H_n^0 = \langle \pi_1, \ldots \pi_{n-1} \rangle$

Hecke monoid at $q = 0$

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Action on $\mathfrak{S}_n$:

$3726145 \cdot s_5 = 3726415$

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Permutation : group

Action on $\mathfrak{S}_n$:

$3726145 \cdot \pi_5 = 3726415$

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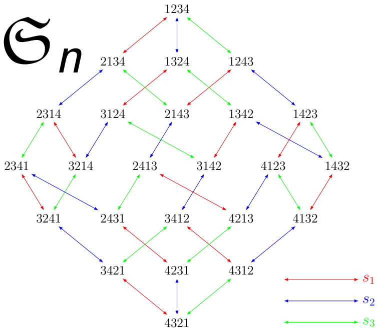
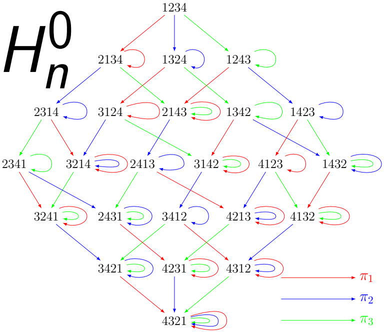
Bubble sort : monoid

Comparison

$$\mathfrak{S}_n \qquad\qquad H_n^0$$

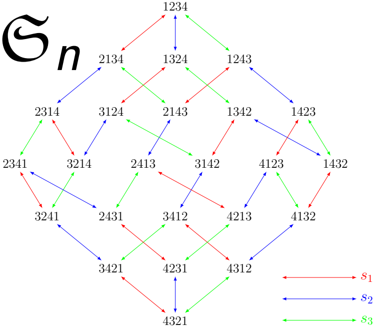
Comparison


$\mathfrak{S}_n$
1234
2134
1324
1243
2314
3124
2143
1342
1423
2341
3214
2413
3142
4123
1432
3241
2431
3412
4213
4132
3421
4231
4312
4321
s_1
s_2
s_3


H_n^0
1234
2134
1324
1243
2314
3124
2143
1342
1423
2341
3214
2413
3142
4123
1432
3241
2431
3412
4213
4132
3421
4231
4312
4321
π_1
π_2
π_3

Comparison

$\mathfrak{S}_n$

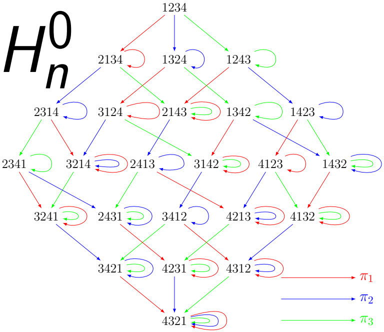


$$s_i^2 = 1$$

$$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$$

$$s_is_j = s_js_i$$

H_n^0



$$\pi_i^2 = \pi_i$$

$$\pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i$$

$$\pi_i\pi_j = \pi_j\pi_i$$

Comparison

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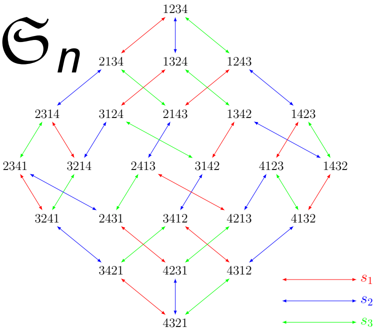
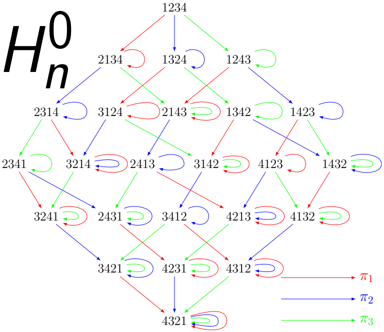


H_n^0



$$s_i^2 = 1$$

$$s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$$

$$s_i s_j = s_j s_i$$

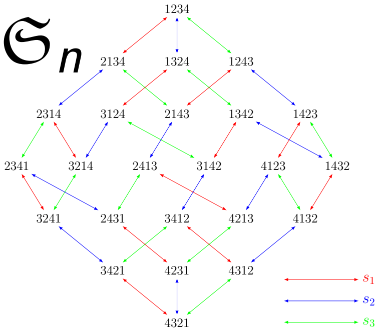
$$\pi_i^2 = \pi_i$$

$$\pi_{i+1} \pi_i \pi_{i+1} = \pi_i \pi_{i+1} \pi_i$$

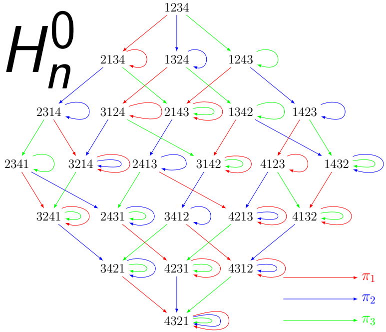
$$\pi_i \pi_j = \pi_j \pi_i$$

Comparison

$\mathfrak{S}_n$



H_n^0



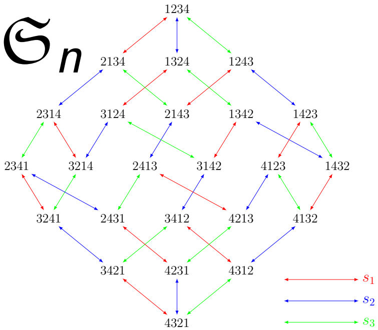
$$s_i^2 = 1 \qquad \pi_i^2 = \pi_i$$

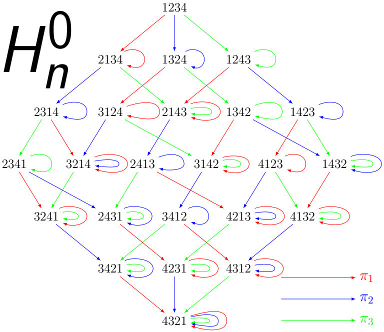
$$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i \qquad \pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i$$

$$s_is_j = s_js_i \qquad \pi_i\pi_j = \pi_j\pi_i$$

Same reduced words (Matsumoto)

Comparison





$$s_i^2 = 1 \qquad\qquad \pi_i^2 = \pi_i$$

$$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i \qquad\qquad \pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i$$

$$s_is_j = s_js_i \qquad\qquad \pi_i\pi_j = \pi_j\pi_i$$

Same reduced words (Matsumoto)

An element of H_n^0 is characterized by its action on the identity

Contents

Rook monoid

Monoid inside $M_n(\mathbb{Z})$.

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Let $\pi_0 := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$

Rook monoid

Monoid inside $M_n(\mathbb{Z})$.

Let $\pi_0 := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$

Rook monoid

$$R_n := \langle \pi_0, s_1, \dots, s_{n-1} \rangle$$

Rook monoid

Monoid inside $M_n(\mathbb{Z})$.

Let $\pi_0 := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$

Rook monoid

$$R_n := \langle \pi_0, s_1, \dots, s_{n-1} \rangle$$

Question:

which matrices are in this monoid?

Notation

Hypothesis:

Rooks are permutations where some letters are replaced by 0.

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Keep the vector notation

Notation

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Keep the vector notation

$$\text{Rook matrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{array}{ccc}
\text{Rook matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\text{Rook vector} & 0 \quad 4 \quad 2 \quad 3 \quad 1 & 0 \quad 3 \quad 0 \quad 4 \quad 1
\end{array}$$

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$$
\begin{array}{lcc}
\text{Rook matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \, & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \, & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \, & 0 & 0 & 0 & 0 \end{pmatrix} \\
\text{Rook vector} & 0_5 \;\; 4 \;\; 2 \;\; 3 \;\; 1 & 0_5 \;\; 3 \;\; 0_2 \;\; 4 \;\; 1
\end{array}
$$

Right action on itself

Operators $\pi_0, s_1, \ldots s_{n-1}$ acting on rook vectors
Action : matrix multiplication

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Action : matrix multiplication

Elementary transpositions $s_1, \dots, s_{n-1}$:
$(r_1 \dots r_n) \cdot s_i = r_1 \dots r_{i-1} r_{i+1} r_i r_{i+2} \dots r_n$
Deletion operator π_0 :
$(r_1 \dots r_n) \cdot \pi_0 = \quad 0 r_2 \dots r_n.$

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Elementary transpositions $s_1, \ldots, s_{n-1}$:
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$(r_1 \ldots r_n) \cdot \pi_0 = \quad 0 r_2 \ldots r_n.$

$\Rightarrow$ Hypothesis stabilized by the generators

<u>Question:</u>

Are any permutations with some letters replaced by 0 in R_n?

Action and algorithm

Example : using coset R_5/R_4

30145

Action and algorithm

Example : using coset R_5/R_4

$$
\begin{array}{c}
3014\textcolor{red}{5} \\
\updownarrow s_4 \\
301\textcolor{red}{5}4
\end{array}
$$

Action and algorithm

Example : using coset R_5/R_4

$$\begin{array}{c} 3014\color{red}{5} \\ \updownarrow s_4 \\ 301\color{red}{5}4 \\ \updownarrow s_3 \\ 30\color{red}{5}14 \end{array}$$

Action and algorithm

Example : using coset R_5/R_4

$$\begin{array}{c} 3014\textcolor{red}{5} \\ \updownarrow s_4 \\ 301\textcolor{red}{5}4 \\ \updownarrow s_3 \\ 30\textcolor{red}{5}14 \\ \updownarrow s_2 \\ 3\textcolor{red}{5}014 \end{array}$$

Action and algorithm

Example : using coset R_5/R_4

$$
\begin{array}{c}
3014\color{red}{5} \\
\updownarrow s_4 \\
301\color{red}{5}4 \\
\updownarrow s_3 \\
30\color{red}{5}14 \\
\updownarrow s_2 \\
3\color{red}{5}014 \\
\updownarrow s_1 \\
\color{red}{5}3014
\end{array}
$$

Action and algorithm

Example : using coset R_5/R_4

$$
\begin{array}{c}
30145 \\
\updownarrow s_4 \\
30154 \\
\updownarrow s_3 \\
30514 \\
\updownarrow s_2 \\
35014 \\
\updownarrow s_1 \\
53014 \\
\downarrow \pi_0 \\
03014
\end{array}
$$

Action and algorithm

Example : using coset R_5/R_4

$$
\begin{array}{c}
3014\color{red}{5} \\
\updownarrow s_4 \\
301\color{red}{5}4 \\
\updownarrow s_3 \\
30\color{red}{5}14 \\
\updownarrow s_2 \\
3\color{red}{5}014 \\
\updownarrow s_1 \\
\color{red}{5}3014 \\
\downarrow \pi_0 \\
\color{red}{0}3014 \\
\updownarrow s_1 \\
3\color{red}{0}014 \circlearrowleft s_2
\end{array}
$$

Action and algorithm

Take a permutation with some letters replaced by 0

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Example : 30240

Action and algorithm

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Index the zeros by the missing letters in decreasing order : 30_5240_1

$12345 \qquad \mathbf{1}_5$

Action and algorithm

Take a permutation with some letters replaced by 0

Example : 30240

Index the zeros by the missing letters in decreasing order : $30_5 240_1$

$$\begin{array}{cc} 12345 & \mathbf{1}_5 \\ \color{red}{0_1}2345 & \cdot\, \pi_0 \end{array}$$

Action and algorithm

Take a permutation with some letters replaced by 0

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

$$\begin{array}{lr} 12345 & \mathbf{1}_5 \\ 0_1 2345 & \cdot\, \pi_0 \\ 2 0_1 345 & \cdot\, s_1 \end{array}$$

Action and algorithm

Take a permutation with some letters replaced by 0

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

$$\begin{array}{lr} 12345 & \mathbf{1}_5 \\ 0_1 2345 & \cdot\, \pi_0 \\ 20_1 345 & \cdot\, s_1 \\ 320_1 45 & \cdot\, s_2 s_1 \end{array}$$

Action and algorithm

Take a permutation with some letters replaced by 0

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

12345	$\mathbf{1}_5$
$0_1 2345$	$\cdot\, \pi_0$
$2 0_1 345$	$\cdot\, s_1$
$3 2 0_1 45$	$\cdot\, s_2 s_1$
$3 2 4 0_1 5$	$\cdot\, s_3$

Action and algorithm

Take a permutation with some letters replaced by 0

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

12345	$\mathbf{1}_5$
$0_1 2345$	$\cdot\, \pi_0$
$20_1 345$	$\cdot\, s_1$
$320_1 45$	$\cdot\, s_2 s_1$
$3240_1 5$	$\cdot\, s_3$
$30_5 240_1$	$\cdot\, s_4 s_3 s_2 s_1 \pi_0 s_1$

Action and algorithm

Take a permutation with some letters replaced by 0

Example : 30240

Index the zeros by the missing letters in decreasing order : $30_5 240_1$

$$\begin{array}{rr} 12345 & \mathbf{1}_5 \\ 0_1 2345 & \cdot\, \pi_0 \\ 20_1 345 & \cdot\, s_1 \\ 320_1 45 & \cdot\, s_2 s_1 \\ 3240_1 5 & \cdot\, s_3 \\ 30_5 240_1 & \cdot\, s_4 s_3 s_2 s_1 \pi_0 s_1 \end{array}$$

Conclusion : $\mathbf{1}_5 \cdot [\pi_0 \cdot s_1 \cdot s_2 s_1 \cdot s_3 \cdot s_4 s_3 s_2 s_1 \pi_0 s_1] = 30240.$

Enumeration

Conclusion:

Rooks are permutations where some letters are replaced by 0.

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Rooks are permutations where some letters are replaced by 0.

That explains "rook" :

$$\begin{array}{ccc}
\text{Rook matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\text{Rook vector} & 0 \quad 4 \quad 2 \quad 3 \quad 1 & 0 \quad 3 \quad 0 \quad 4 \quad 1
\end{array}$$

Enumeration

<u>Conclusion:</u>
Rooks are permutations where some letters are replaced by 0.
That explains "rook" :

$$
\begin{array}{ccc}
\text{Rook matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & \text{♜} & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\text{Rook vector} & \begin{matrix} 0 & 4 & 2 & 3 & 1 \end{matrix} & \begin{matrix} 0 & 3 & 0 & 4 & 1 \end{matrix}
\end{array}
$$

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Rooks are permutations where some letters are replaced by 0.

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$$\begin{array}{lcc}
\text{Rook matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & ♜ \\ 0 & 0 & ♜ & 0 & 0 \\ 0 & 0 & 0 & ♜ & 0 \\ 0 & ♜ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & ♜ \\ 0 & 0 & 0 & 0 & 0 \\ 0 & ♜ & 0 & 0 & 0 \\ 0 & 0 & 0 & ♜ & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\text{Rook vector} & 0\quad 4\quad 2\quad 3\quad 1 & 0\quad 3\quad 0\quad 4\quad 1
\end{array}$$

$$|R_n| = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{k} \cdot (n-k)!$$

Enumeration

<u>Conclusion:</u>

Rooks are permutations where some letters are replaced by 0.
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$$
\begin{array}{ccc}
\text{Rook matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & \text{♜} & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\text{Rook vector} & \begin{matrix} 0 & 4 & 2 & 3 & 1 \end{matrix} & \begin{matrix} 0 & 3 & 0 & 4 & 1 \end{matrix}
\end{array}
$$

$$|R_n| = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{k} \cdot (n-k)! = \sum_{k=0}^{n} \binom{n}{k}^2 \cdot k!$$

Our work

$$\mathfrak{S}_n$$

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$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q)$$

Iwahori

Our work

$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q) \xrightarrow{q=0} H_n^0$$

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$$\begin{array}{ccccc} \mathfrak{S}_n & \xleftarrow{q=1} & \mathcal{H}_n(q) & \xrightarrow{q=0} & H_n^0 \\ \big\downarrow & & & & \\ R_n & & & & \end{array}$$

Iwahori

Our work

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\begin{array}{ccccc}
\mathfrak{S}_n & \xleftarrow{q=1} & \mathcal{H}_n(q) & \xrightarrow{q=0} & H_n^0 \\
\downarrow & & \downarrow & & \\
R_n & \xleftarrow{q=1} & \mathcal{I}_n(q) & &
\end{array}
$$

Iwahori , Solomon.

Our work

$$\begin{tikzcd} \mathfrak{S}_n \arrow[d, hook] & \mathcal{H}_n(q) \arrow[l, "q=1"'] \arrow[r, "q=0"] \arrow[d, hook] & H_n^0 \arrow[d, hook] \\ R_n & \mathcal{I}_n(q) \arrow[l, "q=1"'] \arrow[r, "q=0"] & ?? \end{tikzcd}$$

Iwahori , Solomon.

Our work

$$\begin{array}{ccccc}
\mathfrak{S}_n & \xleftarrow{q=1} & \mathcal{H}_n(q) & \xrightarrow{q=0} & H_n^0 \\
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\end{array}$$

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Definition of R_n^0

Operators $\pi_0, \pi_1, \ldots \pi_{n-1}$ acting on rook vectors

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Bubble sort operators $\pi_1, \dots, \pi_{n-1}$:

$$(r_1 \dots r_n) \cdot \pi_i = \begin{cases} r_1 \dots r_{i-1} r_{i+1} r_i r_{i+2} \dots r_n & \text{if } r_i < r_{i+1}, \\ r_1 \dots r_n & \text{otherwise,} \end{cases}$$

Deletion operator π_0 :

$$(r_1 \dots r_n) \cdot \pi_0 = \quad 0 r_2 \dots r_n.$$

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$$45321 \cdot \pi_1 = 54321$$
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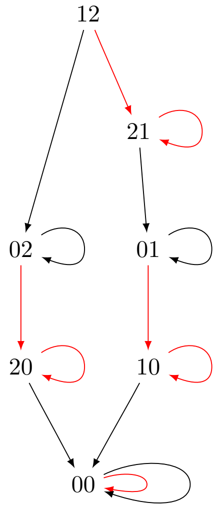

Algorithm

Still works!!

Example : 30240

12345 $\mathbf{1}_5$

Algorithm

Still works!!

Example : 30240

$$
\begin{array}{cc}
12345 & \mathbf{1}_5 \\
0_1 2345 & \cdot\, \pi_0
\end{array}
$$

Algorithm

Still works!!

Example : 30240

$$
\begin{array}{ll}
12345 & \mathbf{1}_5 \\
0_1 2345 & \cdot\, \pi_0 \\
2 0_1 345 & \cdot\, \pi_1
\end{array}
$$

Algorithm

Still works!!

Example : 30240

12345	$\mathbf{1}_5$
$0_1$2345	$\cdot\, \pi_0$
2$0_1$345	$\cdot\, \pi_1$
32$0_1$45	$\cdot\, \pi_2\pi_1$

Algorithm

Still works!!

Example : 30240

12345	$\mathbf{1}_5$
$0_1 2345$	$\cdot\, \pi_0$
$2 0_1 345$	$\cdot\, \pi_1$
$3 2 0_1 45$	$\cdot\, \pi_2 \pi_1$
$3 2 4 0_1 5$	$\cdot\, \pi_3$

Algorithm

Still works!!

Example : 30240

12345	$\mathbf{1}_5$
$0_1 2345$	$\cdot\, \pi_0$
$2 0_1 345$	$\cdot\, \pi_1$
$3 2 0_1 45$	$\cdot\, \pi_2 \pi_1$
$3 2 4 0_1 5$	$\cdot\, \pi_3$
$3 0_5 2 4 0_1$	$\cdot\, \pi_4 \pi_3 \pi_2 \pi_1 \pi_0 \pi_1$

Algorithm

Still works!!

Example : 30240

$$\begin{array}{cr}
12345 & \mathbf{1}_5 \\
0_1 2345 & \cdot\, \pi_0 \\
2 0_1 345 & \cdot\, \pi_1 \\
3 2 0_1 45 & \cdot\, \pi_2\pi_1 \\
3 2 4 0_1 5 & \cdot\, \pi_3 \\
3 0_5 2 4 0_1 & \cdot\, \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1
\end{array}$$

Conclusion : $\mathbf{1}_5 \cdot [\pi_0 \cdot \pi_1 \cdot \pi_2\pi_1 \cdot \pi_3 \cdot \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1] = 30240.$

Algorithm

Still works!!

Example : 30240

$$\begin{array}{cr} 12345 & \mathbf{1}_5 \\ 0_1 2345 & \cdot\, \pi_0 \\ 20_1 345 & \cdot\, \pi_1 \\ 320_1 45 & \cdot\, \pi_2\pi_1 \\ 32 40_1 5 & \cdot\, \pi_3 \\ 30_5 240_1 & \cdot\, \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1 \end{array}$$

Conclusion : $\mathbf{1}_5 \cdot [\pi_0 \cdot \pi_1 \cdot \pi_2\pi_1 \cdot \pi_3 \cdot \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1] = 30240.$

Theorem

$$\begin{array}{rcl} R_n^0 & \longrightarrow & R_n \\ r & \longmapsto & 1 \cdot r \end{array} \quad \textit{is surjective}$$

Presentation

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Corollary: presentation:

$$\begin{aligned} \pi_i^2 &= \pi_i \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} \\ \pi_i\pi_j &= \pi_j\pi_i \\ \pi_1\pi_0\pi_1\pi_0 &= \pi_0\pi_1\pi_0 = \pi_0\pi_1\pi_0\pi_1 \end{aligned}$$

Presentation

Theorem

$$\begin{array}{rcl} R_n^0 & \longrightarrow & R_n \\ r & \longmapsto & 1 \cdot r \end{array} \quad \textit{is bijective}$$

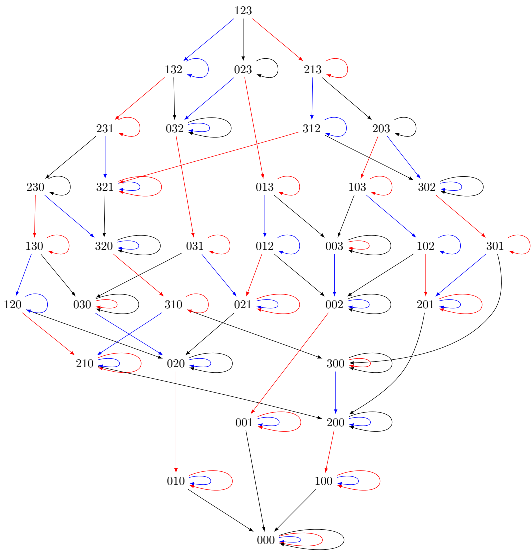
Corollary: presentation:

$$\pi_i^2 = \pi_i$$

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$$

$$\pi_i \pi_j = \pi_j \pi_i$$

$$\pi_1 \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 \pi_1$$



Sorting rooks according to their first zero

$$R(n, k) := \{r \in R_n \mid \mathrm{FZ}(r) = k\}$$

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Example:

$$R(2, 0) = \{00, 01, 02\}, \quad R(2, 1) = \{10, 20\}, \quad R(2, 2) = \{12, 21\}$$

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n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	3	2	2					
3	13	9	6	6				
4	73	52	36	24	24			
5	501	365	260	180	120	120		
6	4051	3006	2190	1560	1080	720	720	
7	37633	28357	21042	15330	10920	7560	5040	5040

Contents

Renner monoid of type A

$$\mathfrak{S}_n = W(A)$$

Weyl Group of type A (Coxeter)

Renner monoid of type A

$$\mathfrak{S}_n = W(A) \quad \longleftrightarrow \quad H_n^0 = H_n^0(A)$$

Weyl Group of type A (Coxeter)
Hecke monoid at $q = 0$ of type A (Hivert, Thiéry)

Renner monoid of type A

$$\begin{array}{ccc} \mathfrak{S}_n = W(A) & \longleftrightarrow & H_n^0 = H_n^0(A) \\ \downarrow & & \\ R_n = \overline{\mathfrak{S}_n} & & \end{array}$$

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Weyl Group of type A (Coxeter)

Hecke monoid at $q = 0$ of type A (Hivert, Thiéry)

Renner monoid of type A (Renner)

0-Renner monoid of type A (G., Hivert)

Renner monoid of type B ??

$$\begin{array}{ccc} W(B) & \longleftrightarrow & H_n^0(B) \\ \downarrow & & \downarrow \\ R_n(B) = \overline{W(B)} & \longleftrightarrow & ?? \end{array}$$

Weyl Group of type B (Coxeter)
Hecke monoid at $q = 0$ of type B (Hivert, Thiéry)
Renner monoid of type B (Renner)
??

Weyl group of type B

$$W(B) = \mathfrak{S}_\ell \wr \mathbb{Z}/2\mathbb{Z}$$

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Example for $\ell = 6$: $\overline{3}\overline{2}46\overline{1}5$

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Example for $\ell = 6$: $\overline{32}46\overline{1}5$

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Let $S_0 := s_\ell$ and for $1 \leq i \leq \ell - 1$, $S_i := s_{\ell-i+1}s_{\ell+i-1}$

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$$E_i := \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \text{ (first } \ell + i \text{ columns are zero)}$$

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$$\forall i, E_i \in \langle E_0, S_0, S_1, \dots, S_{n-1} \rangle$$

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Question: what are the B-rooks?

Actions

Action of S_0:

$$r_{\bar{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\bar{i}} \dots r_{\bar{2}} \color{red}{r_{\bar{1}}} \color{black}{\mid} \color{red}{r_1} \color{black}{r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell}$$

$$r_{\bar{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\bar{i}} \dots r_{\bar{2}} \color{red}{r_1} \color{black}{\mid} \color{red}{r_{\bar{1}}} \color{black}{r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell}$$

Actions

Action of S_1:

$$r_{\overline{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\overline{i}} \dots r_{\overline{2}} r_{\overline{1}} \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

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Actions

Action of S_i:

$$r_{\bar{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\bar{i}} \dots r_{\bar{2}} r_{\bar{1}} \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

$$r_{\bar{\ell}} r_{\overline{\ell-1}} \dots r_{\bar{i}} r_{\overline{i+1}} \dots r_{\bar{2}} r_{\bar{1}} \mid r_1 r_2 \dots r_{i+1} r_i \dots r_{\ell-1} r_\ell$$

Actions

Action of $E := E_0$:

$$r_{\overline{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\overline{i}} \dots r_{\overline{2}} r_{\overline{1}} \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

$$\dots$$

$$0 \dots\dots\dots\dots\dots\dots 0 \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

Actions: example in type B_5

1_{10} :

$$\overline{5}\,\overline{4}\,\overline{3}\,\overline{2}\,\overline{1} \mid 1\,2\,3\,4\,5$$

Actions: example in type B_5

$1_{10} \cdot S_0$:

$$\bar{5}\,\bar{4}\,\bar{3}\,\bar{2}\,\bar{1} \mid 1\,2\,3\,4\,5$$

$$\bar{5}\,\bar{4}\,\bar{3}\,\bar{2}\,1 \mid \bar{1}\,2\,3\,4\,5$$

Actions: example in type B_5

$1_{10} \cdot S_0 S_3$:

$$\overline{5}\,\overline{4}\,\overline{3}\,\overline{2}\,\overline{1} \mid 1\,2\,3\,4\,5$$

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Actions: example in type B_5

$1_{10} \cdot S_0 S_3 E S_0$:

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$$0\,0\,0\,0\,0 \mid \bar{1}\,2\,4\,3\,5$$

$$0\,0\,0\,0\,\bar{1} \mid 0\,2\,4\,3\,5$$

B condition

Definition

$r \in R_{2\ell}$ obeys the B condition iff. it is both:

- (Centrally antisymmetric) r_i and $r_{\bar{i}}$ its mirror image. Then

$$\{r_i, r_{\bar{i}}\} \in \{\{0,0\}, \{0,k\}, \{k,-k\}\}$$

- (Break all pairs) Either no zero, either $\forall i$, i or $-i$ is missing.

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Examples: $\bar{1}3\bar{2} \mid 2\bar{3}1$ $\qquad \bar{3}01 \mid 000$ $\qquad \bar{3}00 \mid 210$
False: $\overline{123} \mid 123$ $\qquad \overline{21}3 \mid 010$ $\qquad 301 \mid 0\bar{3}0$.

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Goal:

Theorem

$r \in R_n(B)$ iff. r obeys the B condition

Algorithm

Let r obeying condition B.
Algorithm to get $s \in R_n(B) \mid 1_{2\ell} \cdot s = r$.

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3. Place the letters in the good order

Explicit Algorithm

$$r = 5\overline{3}001 \mid 0\overline{2}000$$

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$$\overline{5}\overline{1}23\overline{4} \mid 4\overline{3}\overline{2}15$$

Explicit Algorithm

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<u>Step 1:</u> $\overline{54321} \mid 12345 \cdot (S_1 S_2 S_3) = \overline{51432} \mid 23415$

$$\overline{51}23\bar{4} \mid 4\overline{32}15$$

Explicit Algorithm

$$r = 5\bar{3}001 \mid 0\bar{2}000$$

Step 1:

$$\begin{aligned}
\overline{5432{\color{red}1}} \mid {\color{red}1}2345 \cdot (S_1 S_2 S_3) &= \overline{5{\color{red}1}432} \mid 234{\color{red}1}5 \\
\overline{5143{\color{red}2}} \mid {\color{red}2}3415 \cdot (S_0 S_1 S_2) &= \overline{51}{\color{red}2}\overline{43} \mid 34{\color{red}\bar{2}}15 \\
& \overline{51}23\bar{4} \mid 4{\color{red}\overline{32}15}
\end{aligned}$$

Explicit Algorithm

$$r = 5\bar{3}001 \mid 0\bar{2}000$$

<u>Step 1:</u>

$$\begin{array}{rcl}
\bar{5}\bar{4}\bar{3}\bar{2}\bar{1} \mid 12345 \cdot (S_1 S_2 S_3) & = & \bar{5}\bar{1}\bar{4}\bar{3}\bar{2} \mid 23415 \\
\bar{5}\bar{1}\bar{4}\bar{3}\bar{2} \mid 23415 \cdot (S_0 S_1 S_2) & = & \bar{5}\bar{1}2\bar{4}\bar{3} \mid 34\bar{2}15 \\
\bar{5}\bar{1}2\bar{4}\bar{3} \mid 34\bar{2}15 \cdot (S_0 S_1) & = & \bar{5}\bar{1}23\bar{4} \mid 4\bar{3}\bar{2}15
\end{array}$$

Explicit Algorithm

$$r = 5\overline{3}001 \mid 0\overline{2}000$$

Step 1:

$$\overline{54321} \mid 12345 \cdot (S_1 S_2 S_3) \quad = \quad \overline{51432} \mid 23415$$

$$\overline{51432} \mid 23415 \cdot (S_0 S_1 S_2) \quad = \quad \overline{51}2\overline{43} \mid 34\overline{2}15$$

$$\overline{51}2\overline{43} \mid 34\overline{2}15 \cdot (S_0 S_1) \quad = \quad \overline{51}23\overline{4} \mid 4\overline{32}15$$

Step 2:

$$\overline{51}23\overline{4} \mid 4\overline{32}15 \cdot E_1 = 00000 \mid 0\overline{32}15$$

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$$5\bar{3}000 \mid 1\bar{2}000 \cdot S_0 = 5\bar{3}001 \mid 0\bar{2}000 = r$$

$$r = 1_{10} \cdot [(S_1 S_2 S_3)(S_0 S_1 S_2)(S_0 S_1)] \cdot E_1 \cdot [(S_4 S_3 S_2 S_1 S_0 S_1 S_2 S_3 S_4)(S_1 S_0 S_1 S_2 S_3)(S_1 S_2)(S_1)(S_0)]$$

Bilan and enumeration

Theorem

$r \in R_n(B)$ iff. r obeys the B condition

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$$|R_n(B)| = 2^\ell \ell! + \sum_{k=0}^{\ell} \binom{\ell}{k} \binom{\ell}{k} 2^{\ell-k} 2^{\ell-k} (\ell - k)!$$

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$$= 2^\ell \ell! + \sum_{k=0}^{\ell} \binom{\ell}{k}^2 4^k k!$$

0-Renner of type B

$R^0_\ell(B) \subset R^0_{2\ell}$
Operators $E, \Pi_0, \Pi_1, \ldots \Pi_{\ell-1}$ acting on B-rook vectors

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$(r_{\overline{\ell}} \ldots r_\ell) \cdot E = 0 \ldots 0 r_1 \ldots r_\ell$.

0-Renner of type B

$R_\ell^0(B) \subset R_{2\ell}^0$
Operators $E, \Pi_0, \Pi_1, \dots \Pi_{\ell-1}$ acting on B-rook vectors

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Deletion operator E:
$(r_{\overline{\ell}} \dots r_\ell) \cdot E = 0 \dots 0 r_1 \dots r_\ell$.

$$R_\ell^0(B) = \langle E, \Pi_0, \Pi_1, \dots \Pi_{\ell-1} \rangle$$

Bijection

$$\begin{array}{ccc} R_n^0(B) & \longrightarrow & R_n(B) \\ r & \longmapsto & 1 \cdot r \end{array}$$

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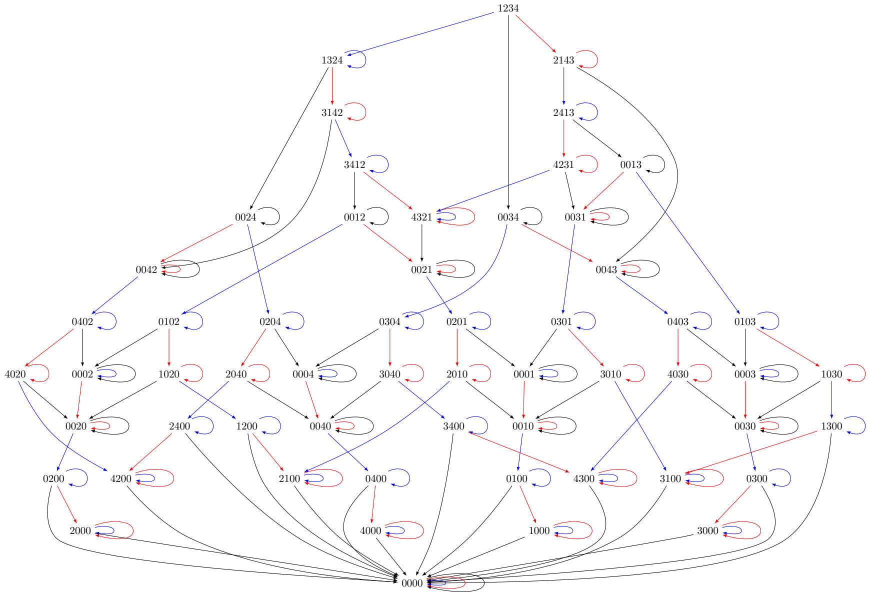
Theorem

$$\begin{array}{rcl} R_n^0(B) & \longrightarrow & R_n(B) \\ r & \longmapsto & 1 \cdot r \end{array}$$ *is bijective*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Presentation

The algorithm gives us for all $r \in R_n^0(B)$:

$$r = \Pi E_j \Pi'$$

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This decomposition is unique

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Π and Π' are reduced elements of $H_n^0(B)$
Π has an increasing form: it is a grassmanian element
This decomposition is unique
We get a presentation

Presentation

$$P_i := \Pi_1 \dots \Pi_i \Pi_1 \dots \Pi_{i-1} \Pi_1 \dots \Pi_{i-2} \dots \Pi_1 \Pi_2 \Pi_1$$

$$\begin{array}{llcll}
(1) & \Pi_i^2 & = & \Pi_i & 0 \le i \le \ell - 1; \\
(2) & \Pi_i \Pi_j & = & \Pi_j \Pi_i, & 0 \le i, j \le \ell - 1 \text{ and } |i - j| \ge 2; \\
(3) & \Pi_i \Pi_{i+1} \Pi_i & = & \Pi_{i+1} \Pi_i \Pi_{i+1}, & 1 \le i \le \ell - 2; \\
(4) & \Pi_1 \Pi_0 \Pi_1 \Pi_0 & = & \Pi_0 \Pi_1 \Pi_0 \Pi_1, & \\
(5) & E_j \Pi_i & = & \Pi_i E_j = E_j, & 0 \le i < j \le \ell; \\
(6) & E_j \Pi_i & = & \Pi_i E_j, & 0 \le j < i \le \ell - 1; \\
(7) & E_i E_j & = & E_j E_i = E_{\max(i,j)}, & 0 \le i, j \le \ell; \\
(8) & E_i \Pi_i E_i & = & E_{i+1}, & 0 \le i \le \ell - 1; \\
(9) & E_0 P_i E_0 & = & E_{i+1}, & 0 \le i \le \ell - 1.
\end{array}$$

Results

In type A, B, D:

- Description of elements of $R_n(T)$
- Definition of $R_n^0(T)$
- Isomorphism with $R_n(T)$
- Canonical reduced words, presentation
- $\mathcal{J}$-triviality
- Simple and indecomposable modules, projectivity over $H_n^0(T)$, quiver

Only in type A

- Explicit order on elements of R_n and R_n^0, lattice, permutohedre
- Induction and restriction of modules

THANK YOU FOR YOUR OUTSTANDING ATTENTION!!

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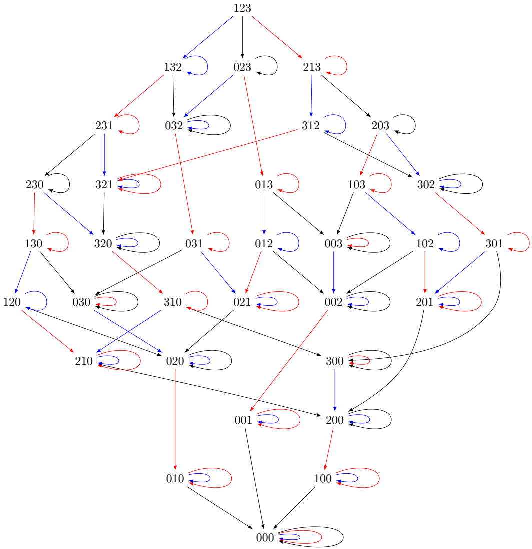
Contents

R_3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Small changes in type D

Action of S_1^e:

$$r_{\overline{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\overline{i}} \dots r_{\overline{2}} r_{\overline{1}} \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

$$r_{\overline{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\overline{i}} \dots r_{\overline{1}} r_{\overline{2}} \mid r_2 r_1 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

Small changes in type D

Action of S_1^f:

$$r_{\overline{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\overline{i}} \dots r_{\overline{2}} r_{\overline{1}} \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

$$r_{\overline{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\overline{i}} \dots r_1 r_2 \mid r_{\overline{2}} r_{\overline{1}} \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

Small changes in type D

Action of E:

$$r_{\overline{\ell}}\, r_{\overline{\ell-1}} \dots r_{\overline{i+1}}\, r_{\overline{i}} \dots r_{\overline{2}}\, r_{\overline{1}} \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

$$\dots$$

$$0 \;\dots\dots\dots\dots\dots\; 00 \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

Small changes in type D

Action of F:

$$r_{\overline{\ell}} r_{\overline{\ell-1}} \dots r_{\overline{i+1}} r_{\overline{i}} \dots r_{\overline{2}} r_{\overline{1}} \mid r_1 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

$$\dots$$

$$0 \;\dots\dots\dots\dots\dots\dots\; 0 r_{\overline{1}} \mid 0 r_2 \dots r_i r_{i+1} \dots r_{\ell-1} r_\ell$$

D condition

$$R_n(D) = \langle S_1^e, S_1^f, S_2, \ldots, S_{\ell-1}, E, F \rangle$$

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Definition

$r \in R_{2\ell}$ obeys the D condition iff. it is both:

- (B rook) obeys the B condition
- (Parity) if it has no 0, even number of positive elements in first half. If ℓ zeros, same in its antisymmetric reconstruction.

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Definition

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- (B rook) obeys the B condition
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Examples: $1\overline{3}2 \mid \overline{2}3\overline{1}$ $\quad$ $100 \mid 200$ $\quad$ $100 \mid \overline{2}30$

False: $1\overline{32} \mid 23\overline{1}$ $\quad$ $100 \mid \overline{23}0$

Enumeration

Theorem

$r \in R_n(D)$ iff. r obeys the D condition

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$$
\begin{aligned}
|R_n(D)| &= \frac{2^\ell \ell!}{2} + \frac{\ell! 4^\ell}{2} + \sum_{k=0}^{\ell-1} 4^k \binom{\ell}{k}^2 k! \\
&= \ell! 2^{\ell-1}(1 - 2^\ell) + \sum_{k=0}^{\ell} 4^k \binom{\ell}{k}^2 k!
\end{aligned}
$$

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$$= \ell! 2^{\ell-1}(1 - 2^\ell) + \sum_{k=0}^{\ell} 4^k \binom{\ell}{k}^2 k!$$

We can also define $R_n^0(D)$, same properties.