Fighting Fish:

enumerative properties

Enrica Duchi

IRIF, Université Paris Diderot

Veronica Guerrini
& Simone Rinaldi

DIISM, Università di Siena

Gilles Schaeffer

LIX, CNRS and École Polytechnique

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Summary of the talk

Fighting fish, a new combinatorial model
of discrete branching surfaces

Exact counting formulas for fighting fish

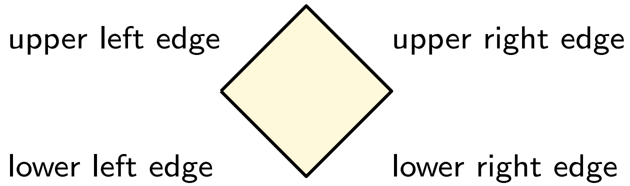
A decomposition for fighting fish

Fighting fish VS classical combinatorial structures
a bijective challenge...

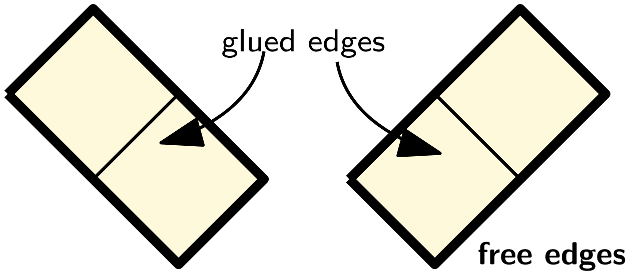
Fighting fish, definition

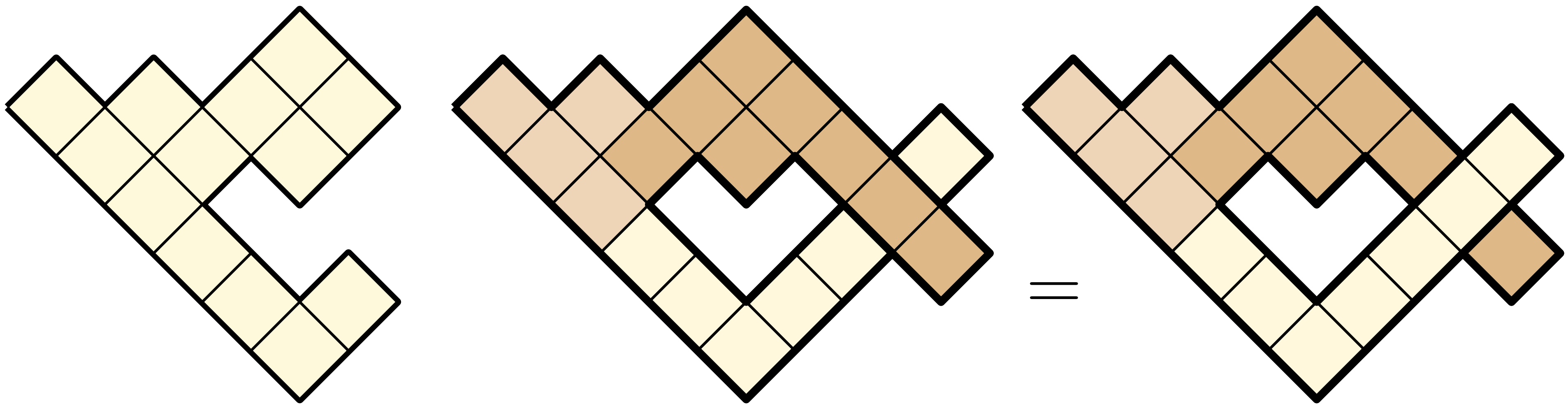
Cells

45^o tilted unit square
(of thin paper or cloth)



Build surface by gluing cells along edges in a coherent way: upper left with lower right or lower left with upper right.

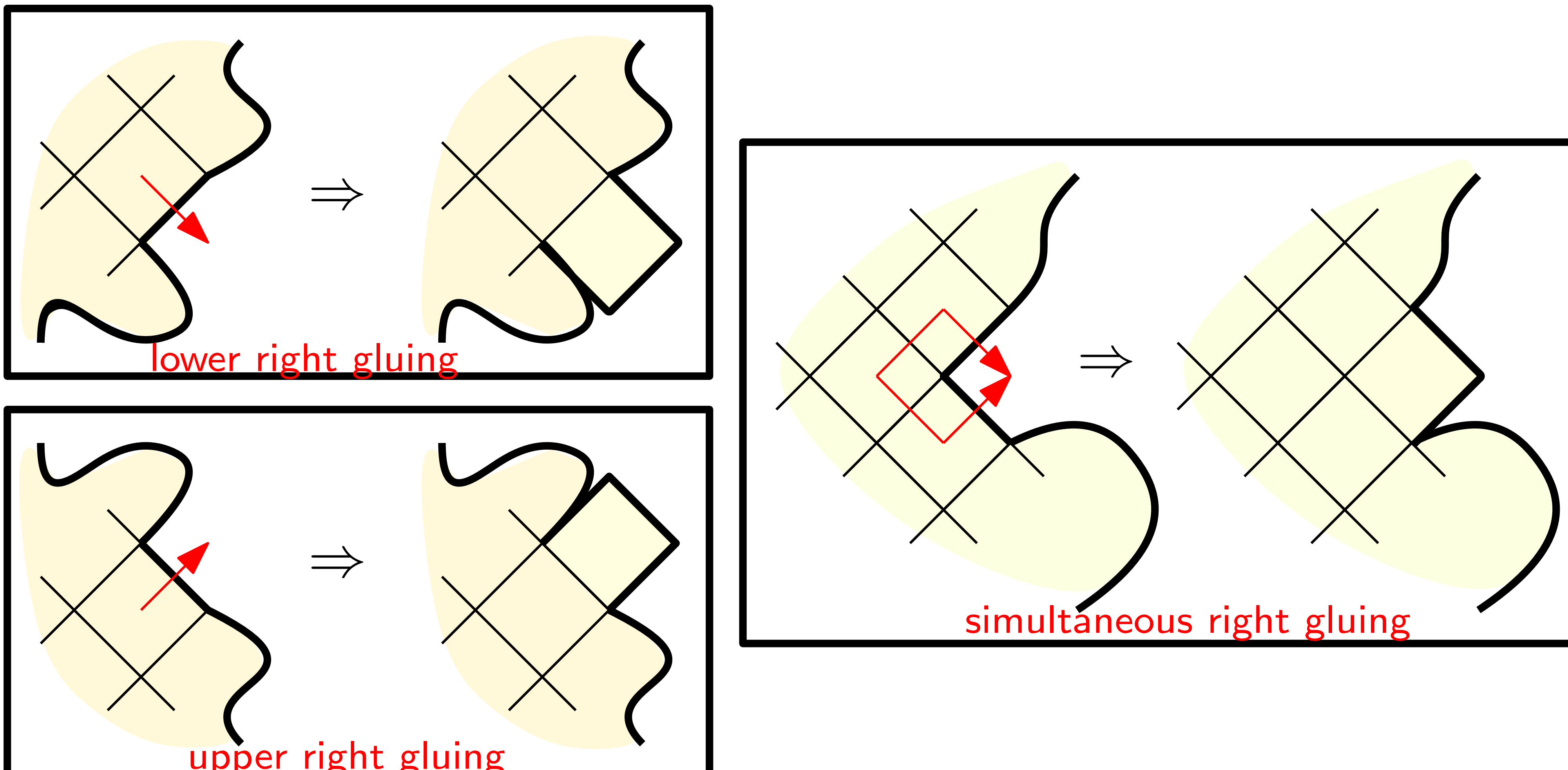




These objects do not necessarily fit in the plane so my pictures are projections of the actual surfaces: Apparently overlapping cells are in fact independant.

Fighting fish, definition

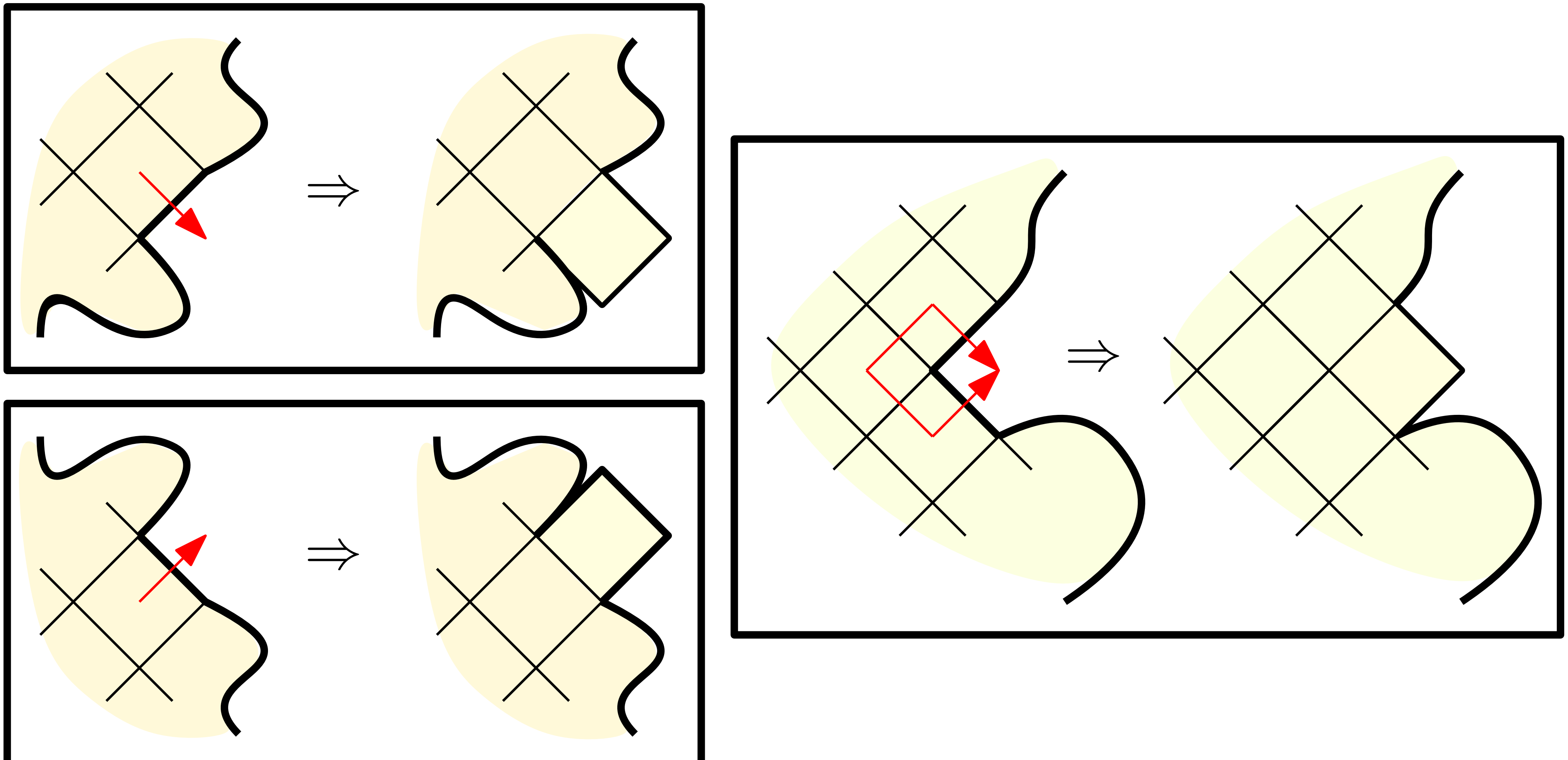
Directed cell aggregation. Restrict to only three legal ways to add cells: by lower right gluing, upper right gluing, or simultaneous lower and upper right gluings from adjacent free edges.



Fighting fish, definition

Lemma. Single cell + aggregations
$\Rightarrow$ a simply connected surface

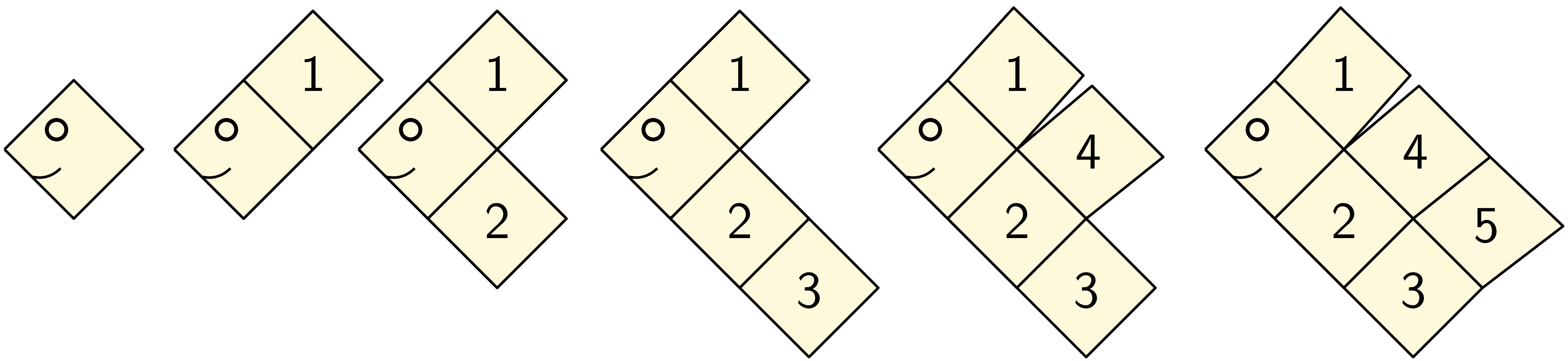
Remark. Such surfaces can be recovered from their boundary walk.



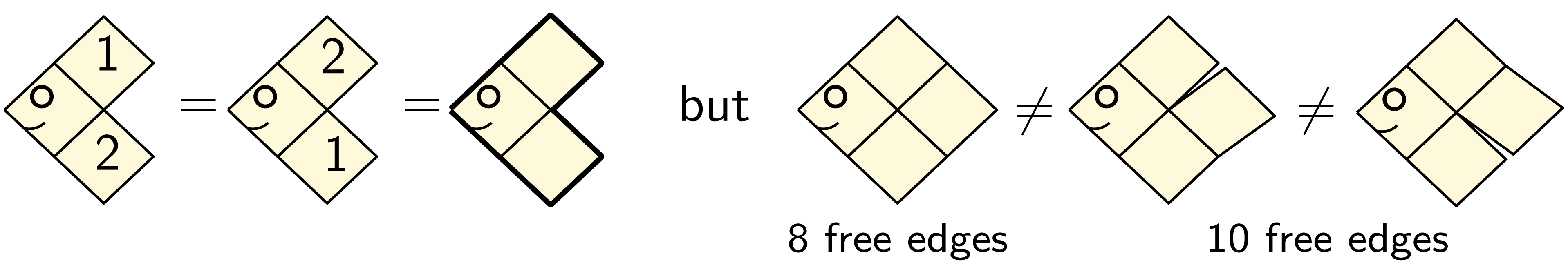
Fighting fish, definition

Fighting fish

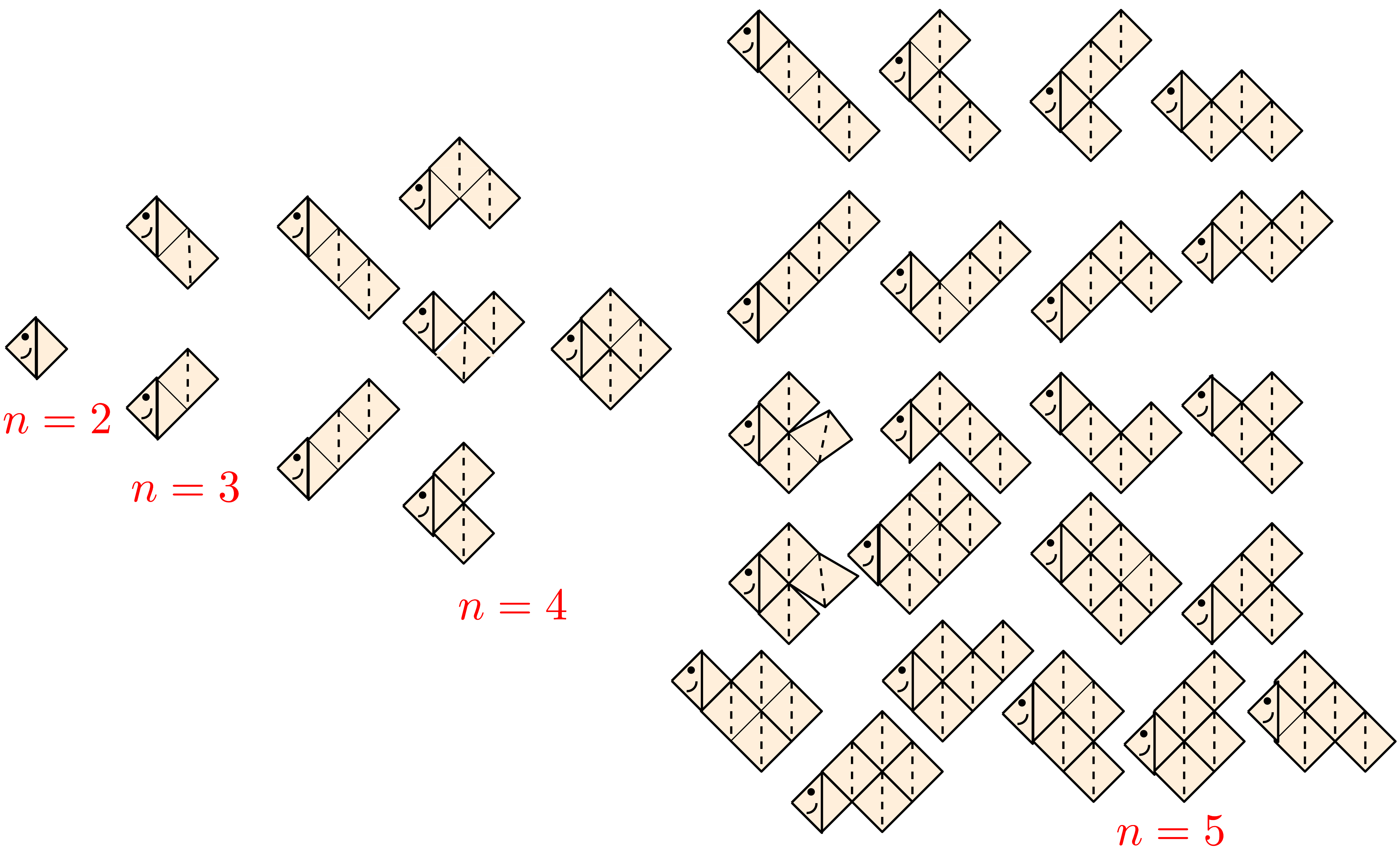
A fighting fish is a surface that can be obtained from a single cell by a sequence of directed cell agregations.



We are interested only in the resulting surface, not in the aggregation order (but type of aggregation matters)

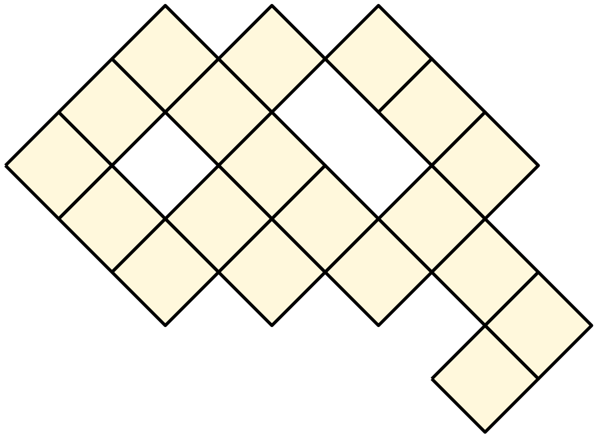


Small fighting fish



Fighting fish versus polyominoes

Polyomino = edge-connected set of cells **of the planar square lattice**



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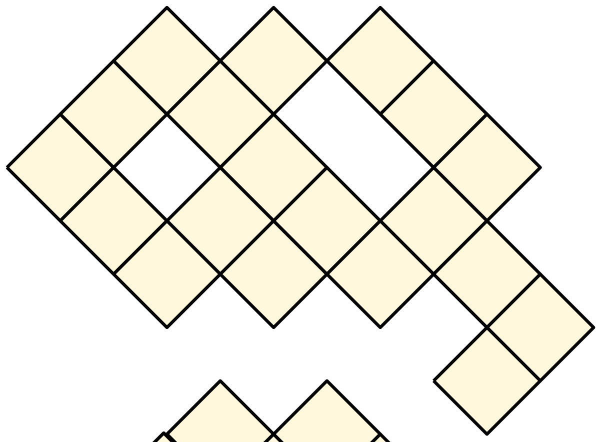

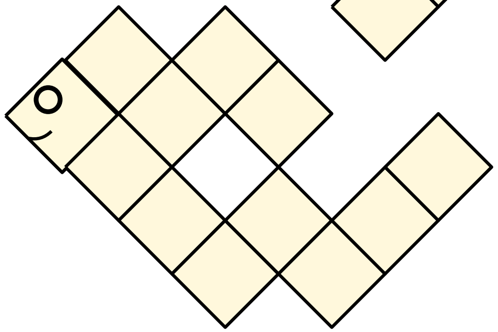

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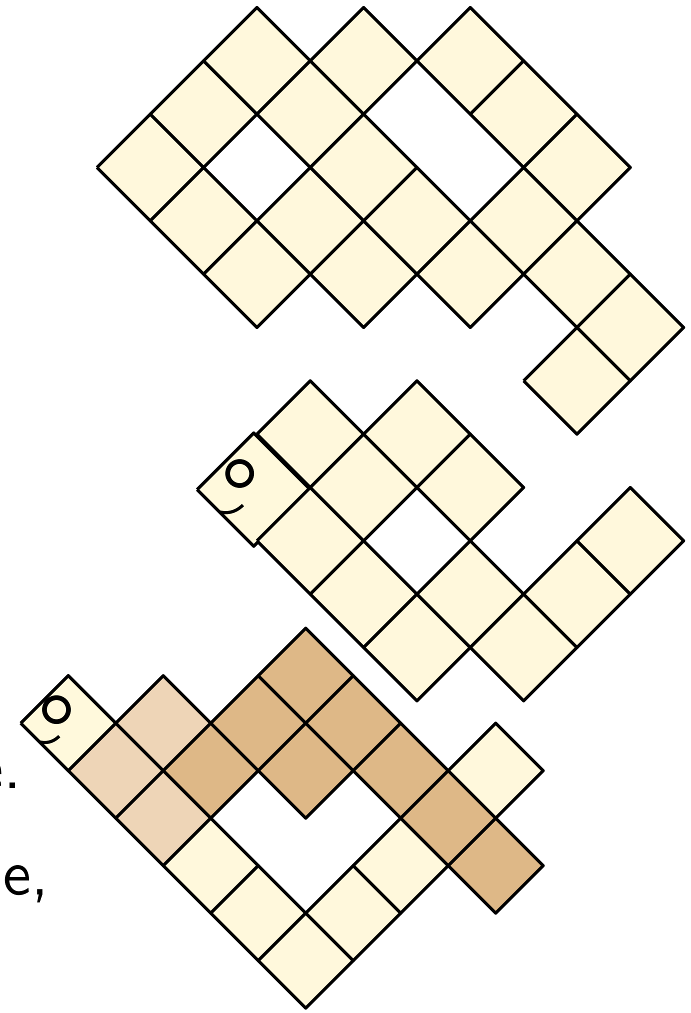
A fighting fish is a directed polyomino **iff** its projection in the plane is injective.

$\Rightarrow$ fighting fish do not all fit in the plane, *ie* they are not all polyominoes.



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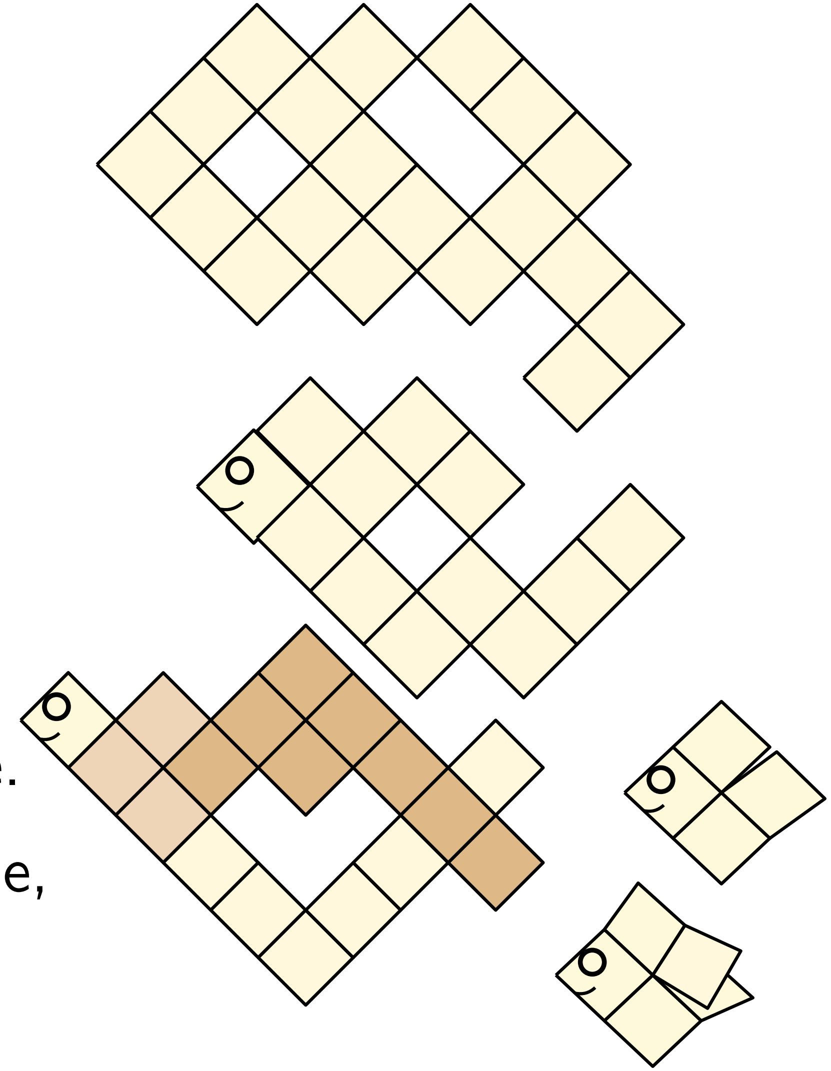

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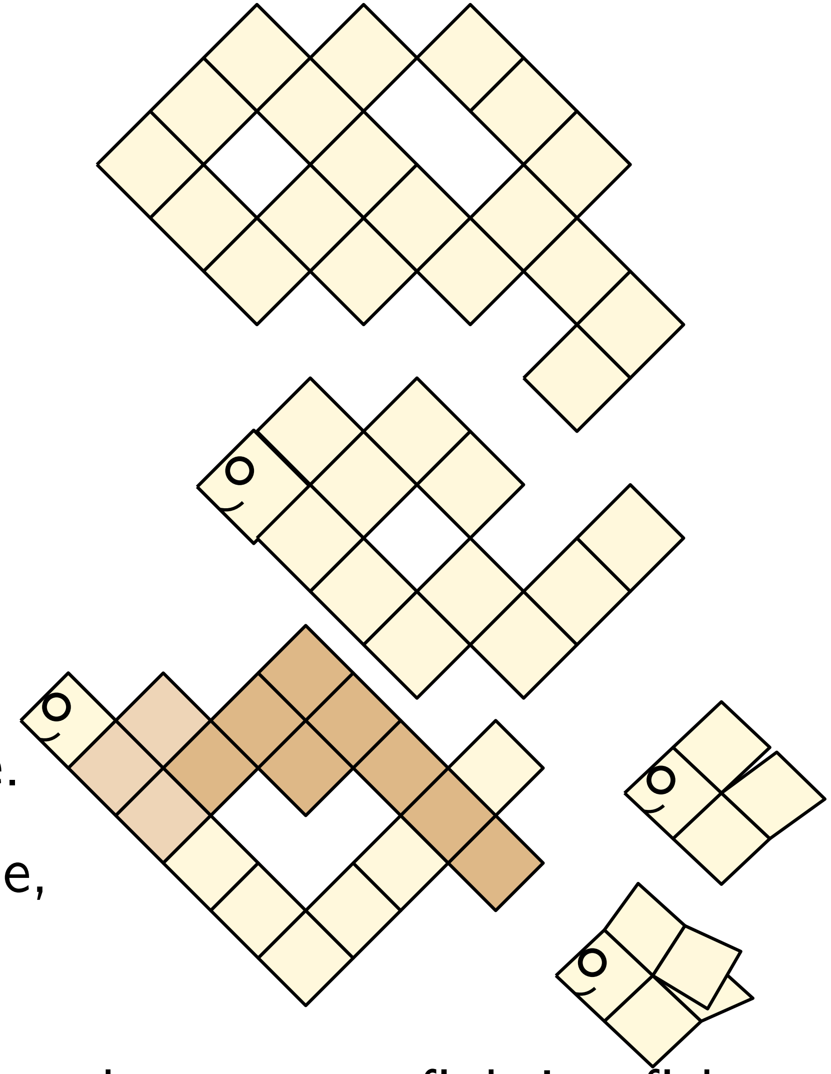
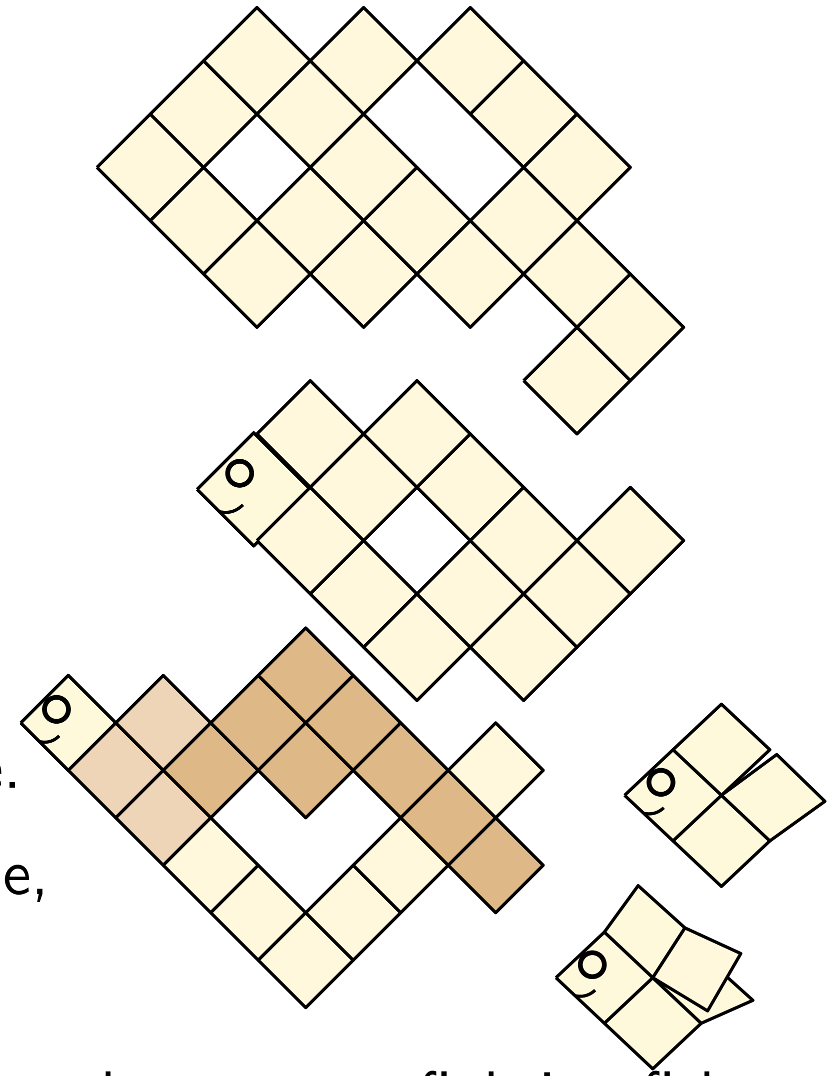


Conversely there are directed polyominoes that are not fighting fish:

Fighting fish versus polyominoes



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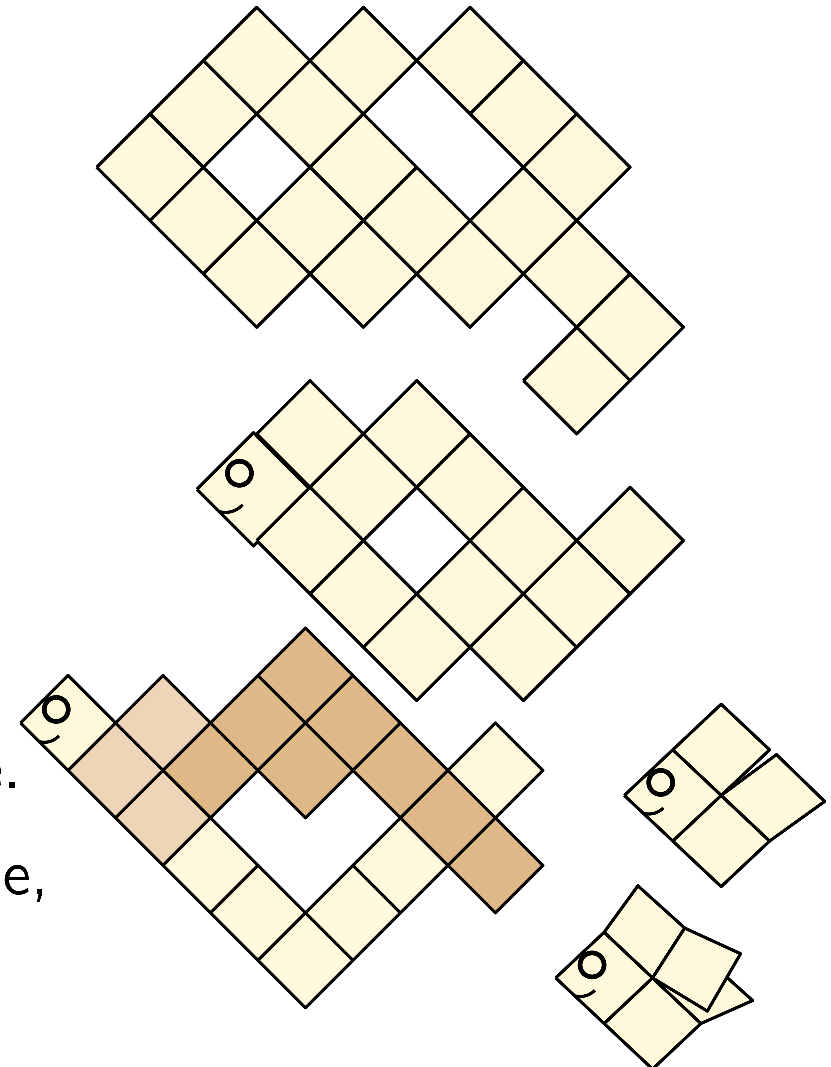

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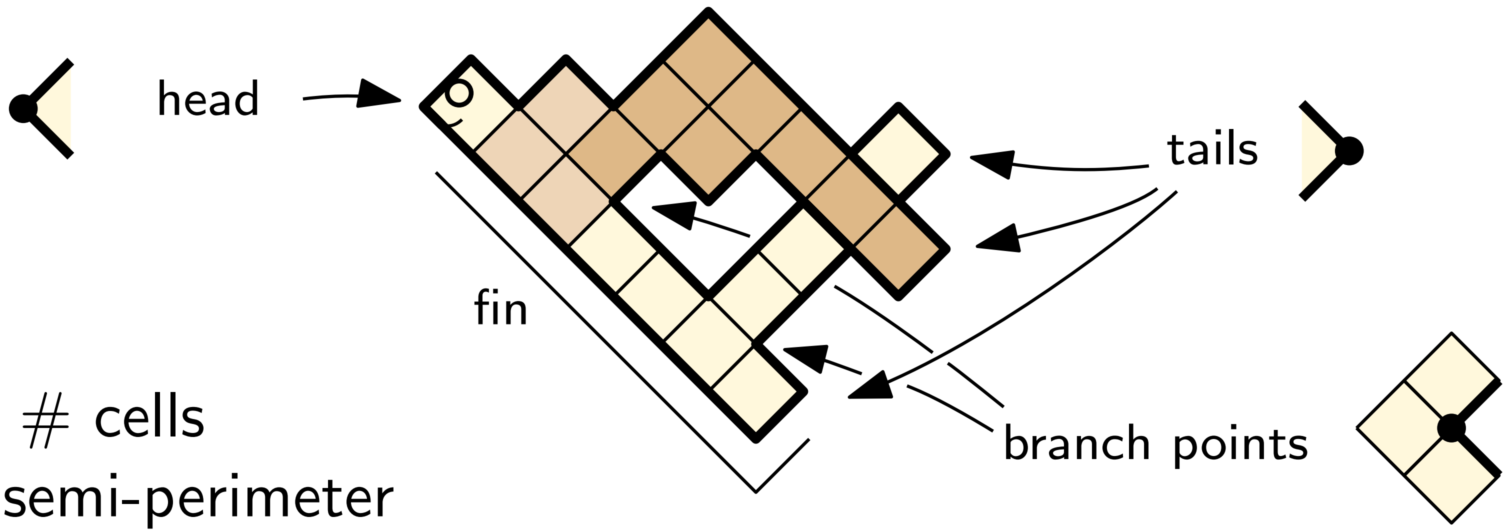
Parameters of fighting fish



Area $=$ # cells
Size $=$ semi-perimeter
$= \#\{$upper free edges$\}$
$= \#\{$upper left free edges$\} + \#\{$upper right free edges$\}$

The fin length $= \#\{$ lower free edges from head to first tail $\}$

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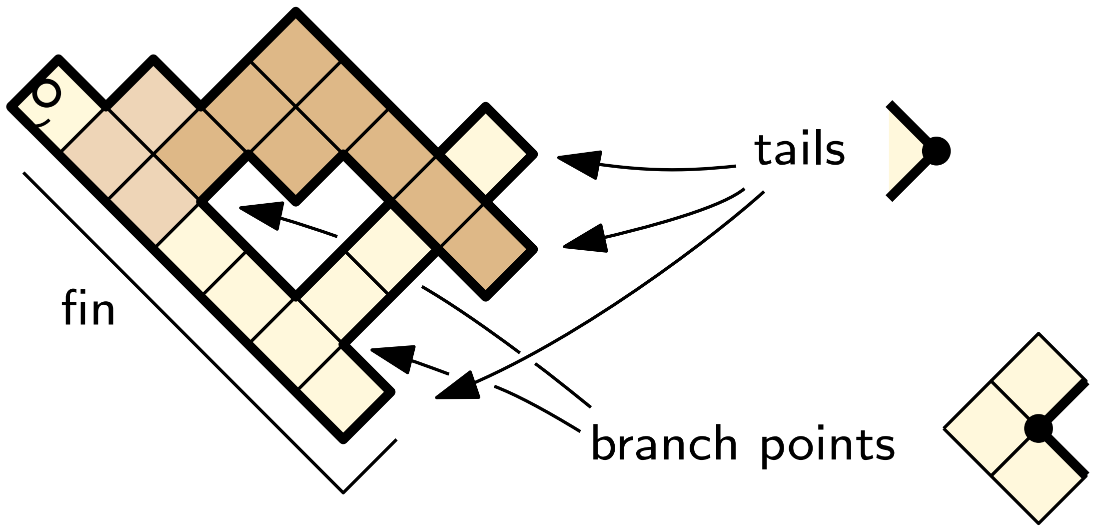

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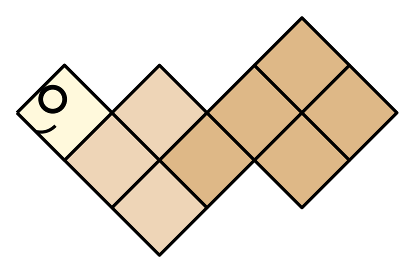
Fighting fish with exactly 1 tail



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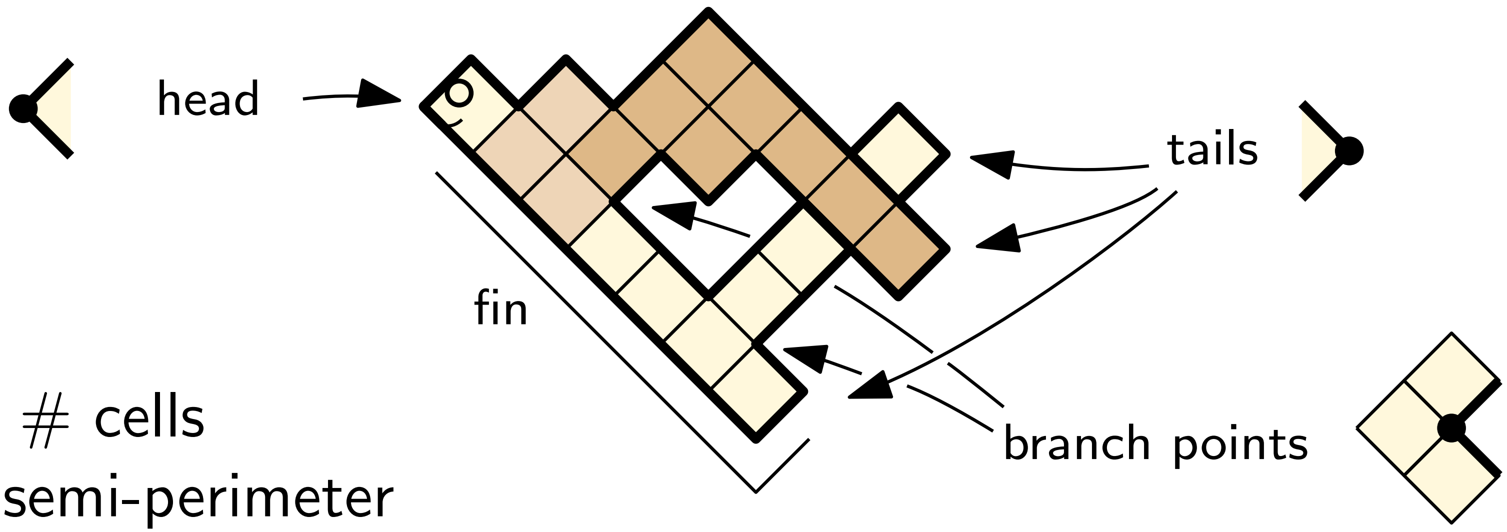

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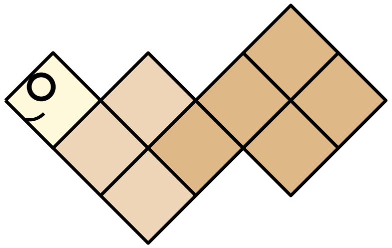
Fighting fish with exactly 1 tail

$=$ parallelogram polyominoes

aka staircase polygons



in this case, fin length $=$ semi-perimeter

Enumerative results

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fighting fish with 1 tail

Theorem (folklore)

$$\#\left\{\begin{array}{c}\text{parallelogram polyominos}\\ \text{with semi-perimeter } n+1\end{array}\right\} = \frac{1}{2n+1}\binom{2n}{n}$$

$$\#\left\{\begin{array}{c}\text{parallelogram polyominos with}\\ i \text{ top left and } j \text{ top right edges}\end{array}\right\} = \tfrac{1}{i+j-1}\tbinom{i+j-1}{i}\tbinom{i+j-1}{j}$$

Enumerative results

fighting fish with 1 tail

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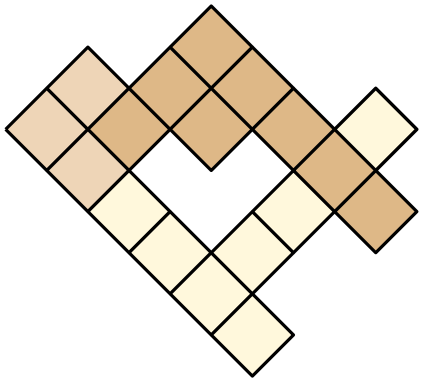
Theorem (D., Guerrini, Rinaldi, Schaeffer, 2016)

$$\#\left\{\begin{array}{c}\text{fighting fish}\\ \text{with semi-perimeter } n+1\end{array}\right\} = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

$$\#\left\{\begin{array}{c}\text{fighting fish with}\\ i \text{ top left and } j \text{ top right edges}\end{array}\right\} = \frac{1}{(2i+j-1)(2j+i-1)}\binom{2i+j-1}{i}\binom{2j+i-1}{j}$$

Fighting fish as random branching surfaces

Let F_n be a fighting fish taken uniformly at random among all fighting fish of size n.



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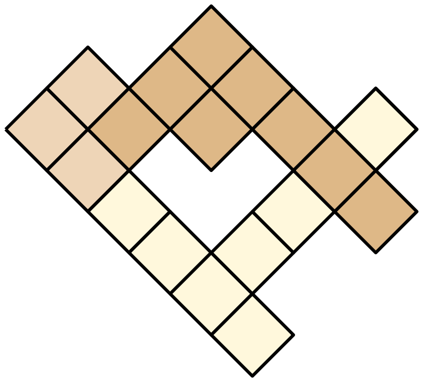

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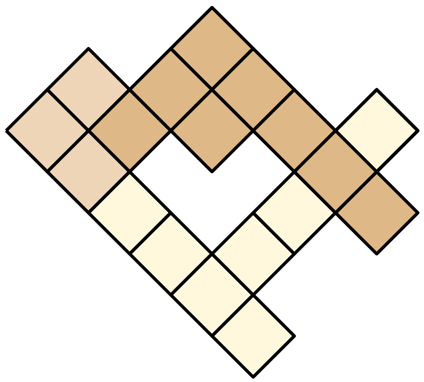

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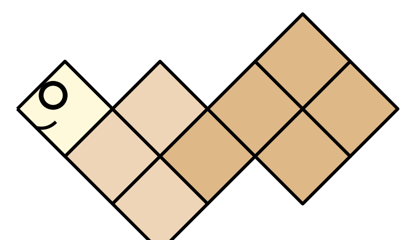
Compare to the known expected area $n^{3/2}$ of random parallelogram polyominoes of size n



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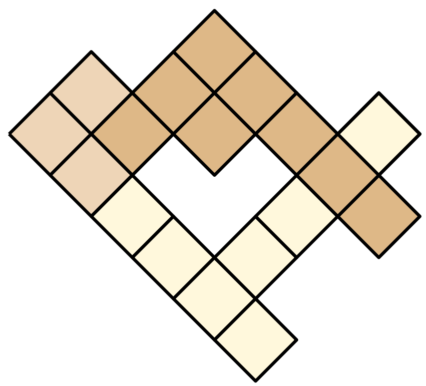

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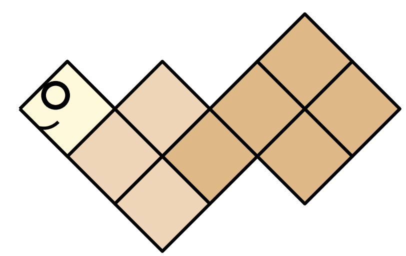

Uniform random fighting fish of size n gives a new model of random branching surfaces with original features.

Fish tails

We start by giving the definition of a slightly more general class: **Fighting fish tails.**

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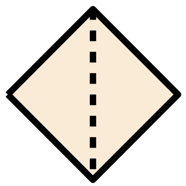
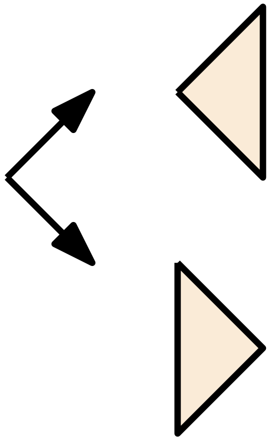
A cell  is made up by two scales 

Left scale

Right scale

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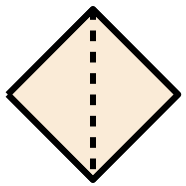
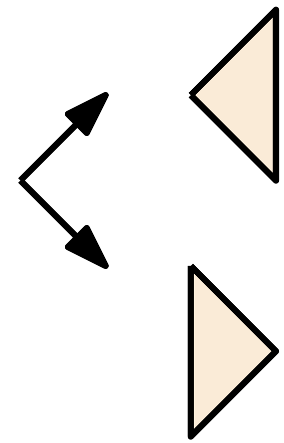
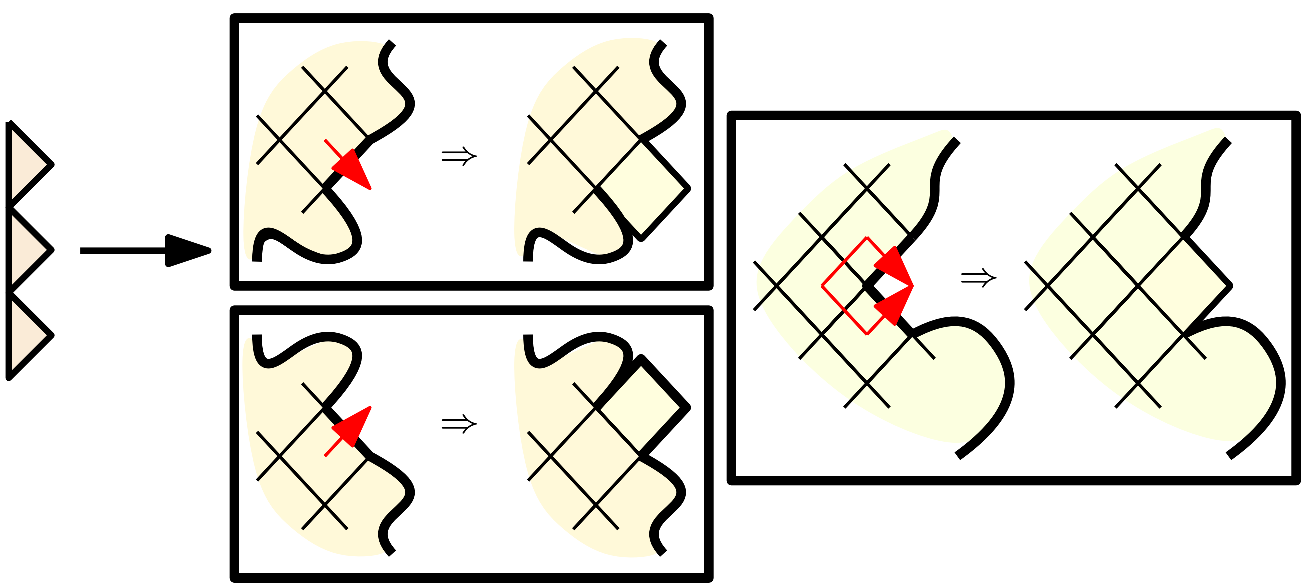
A cell  is made up by two scales 

Left scale

Right scale

A **fish tail** is a surface that can be obtained from a strip of right scales by a sequence of directed cell agregations.



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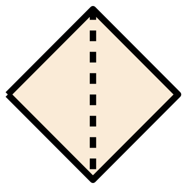
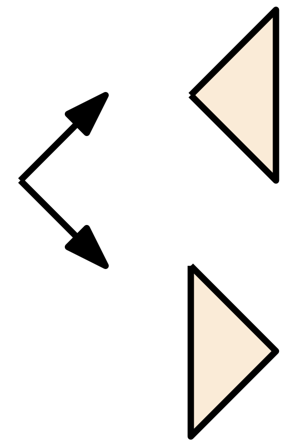
A cell  is made up by two scales 

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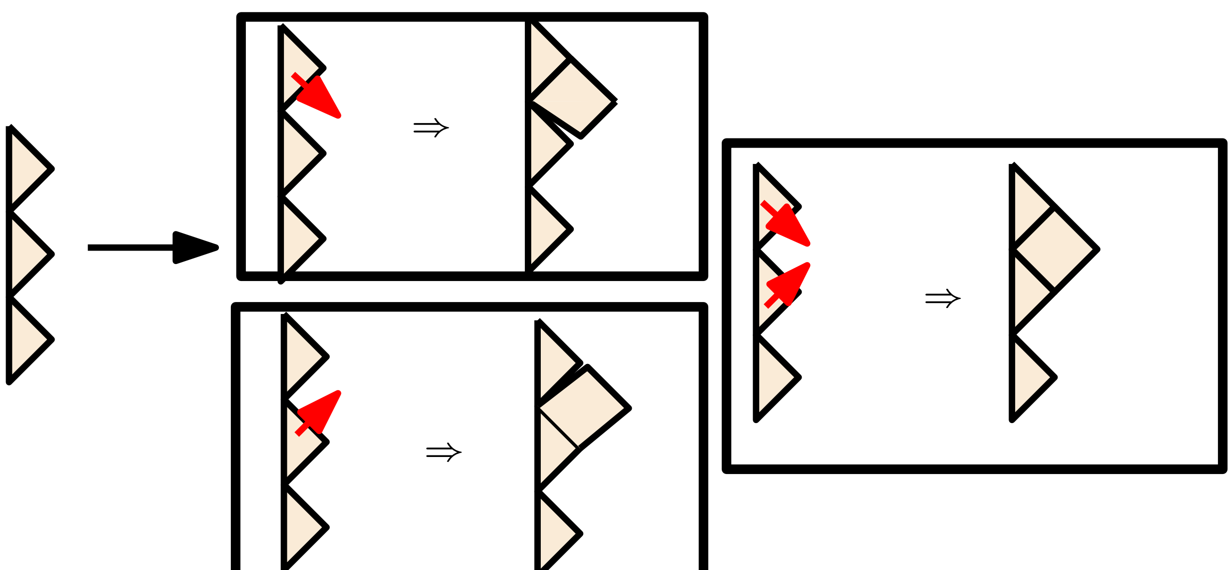

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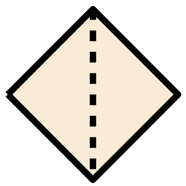
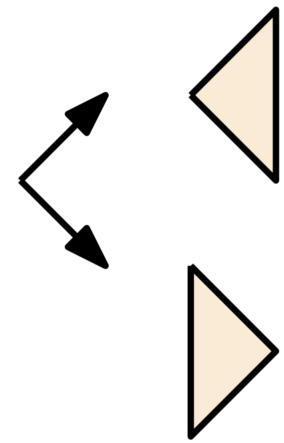
A cell  is made up by two scales



Left scale

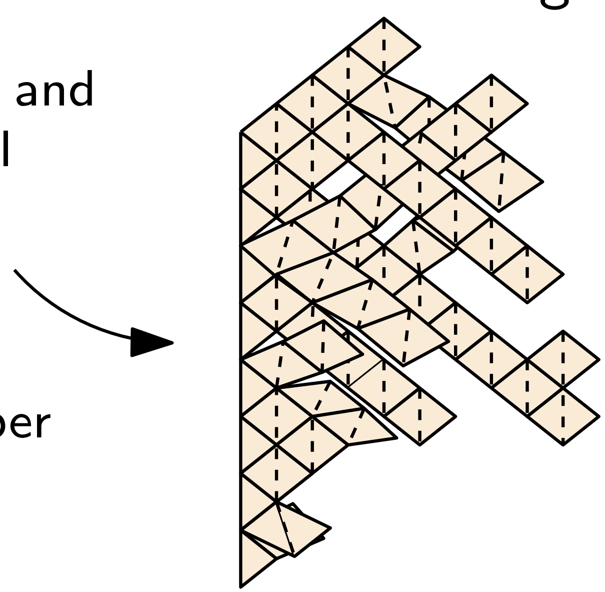
Right scale

A **fish tail** is a surface that can be obtained from a strip of right scales by a sequence of directed cell agregations.

area= the number of left and right scales in the fish tail

height= the number of right scales in the strip

size= the number of upper left and right free edges



A recursive decomposition

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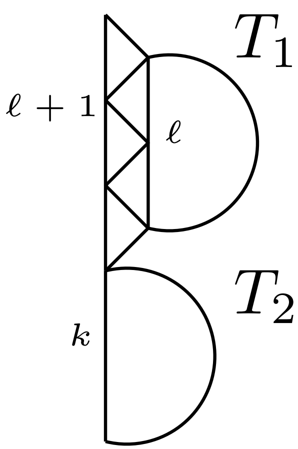
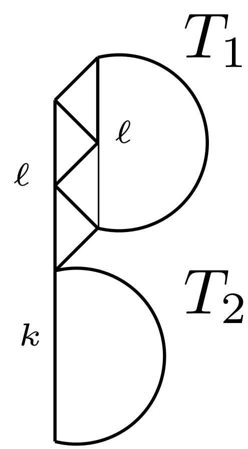
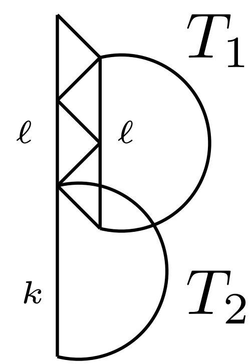
The empty fish is the unique fish tail with height 0.



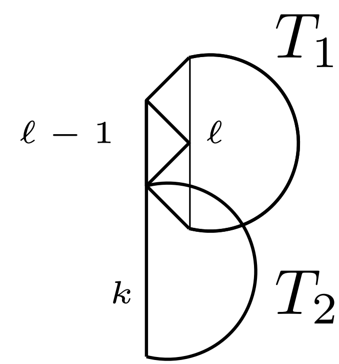
Operation u



Operation h



Operation h'



Operation d

A recursive decomposition

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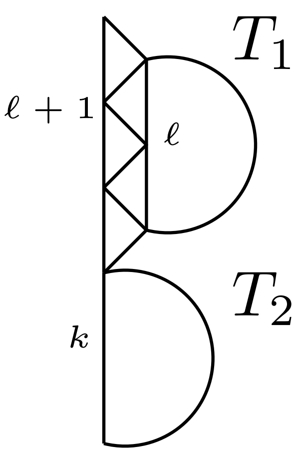

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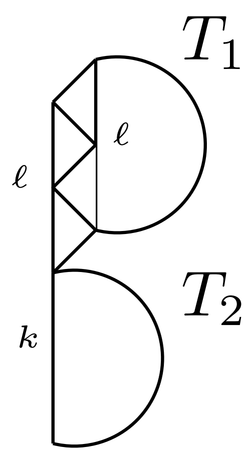

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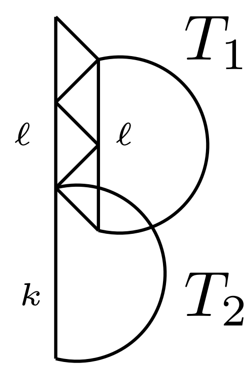

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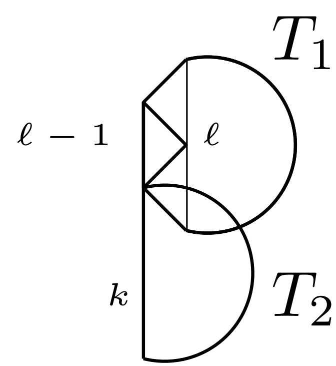
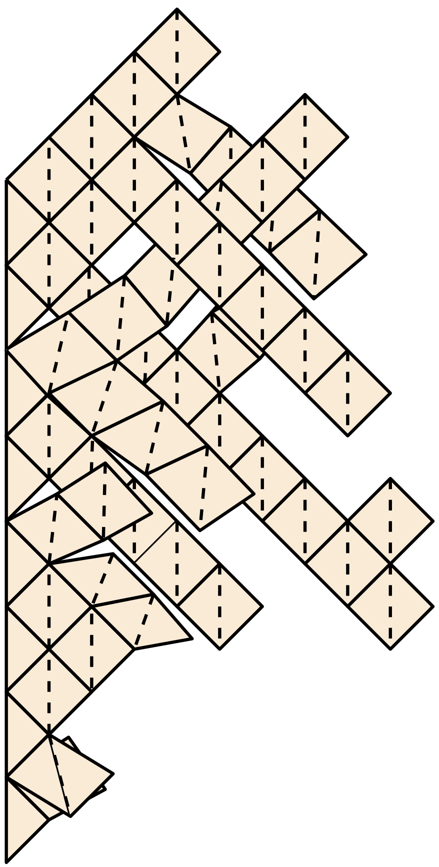


Operation d

Every fish tail can be obtained in a unique way using operations u, h, h', d.

A recursive definition



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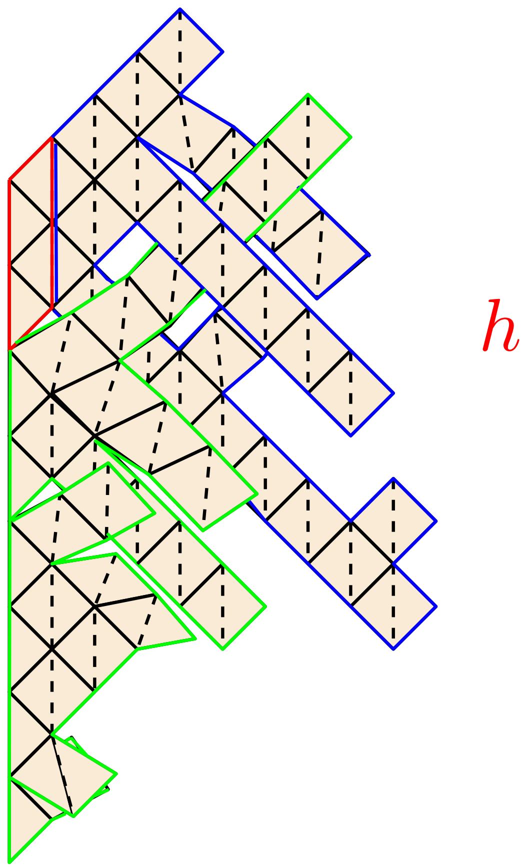

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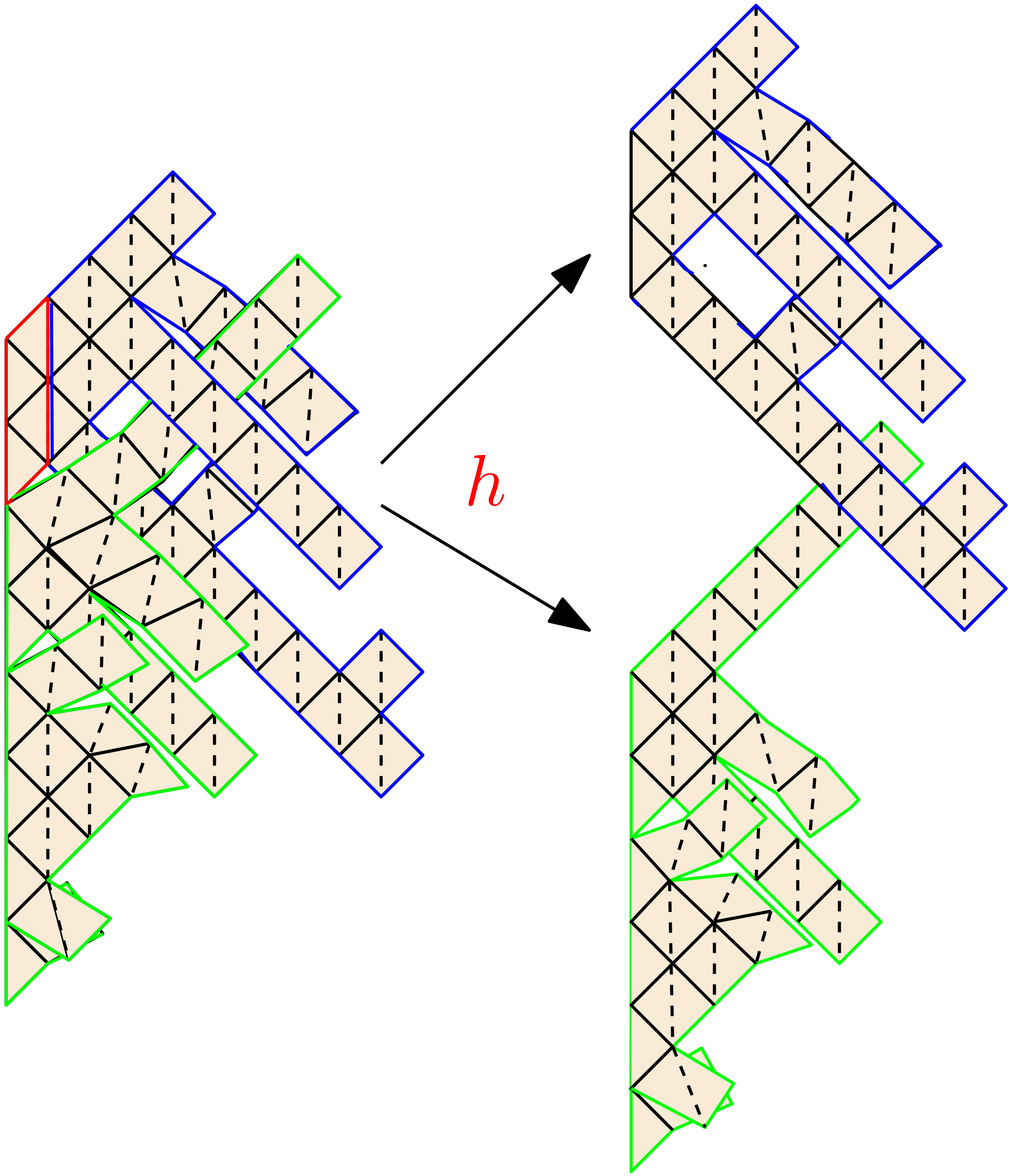

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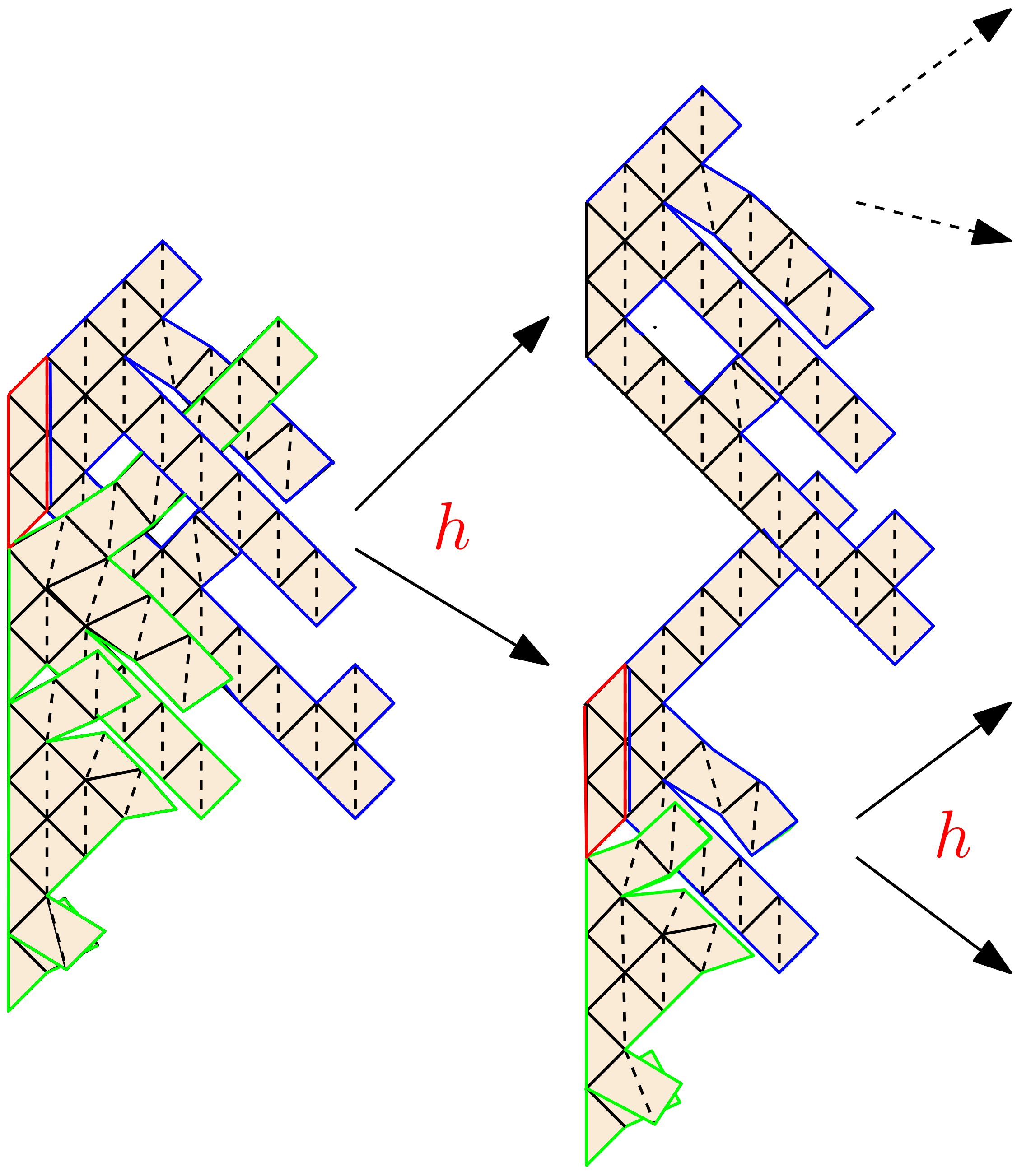

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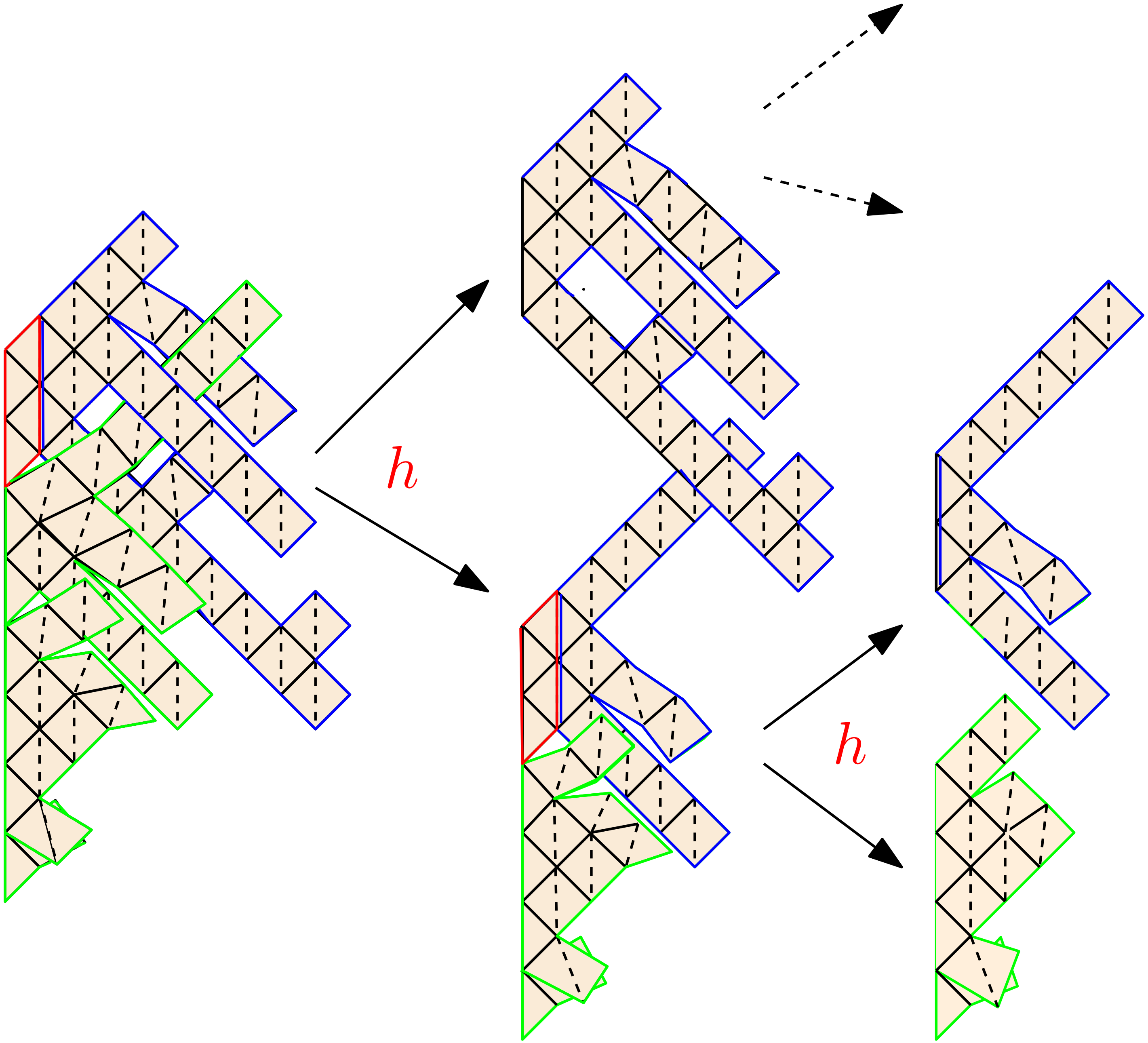

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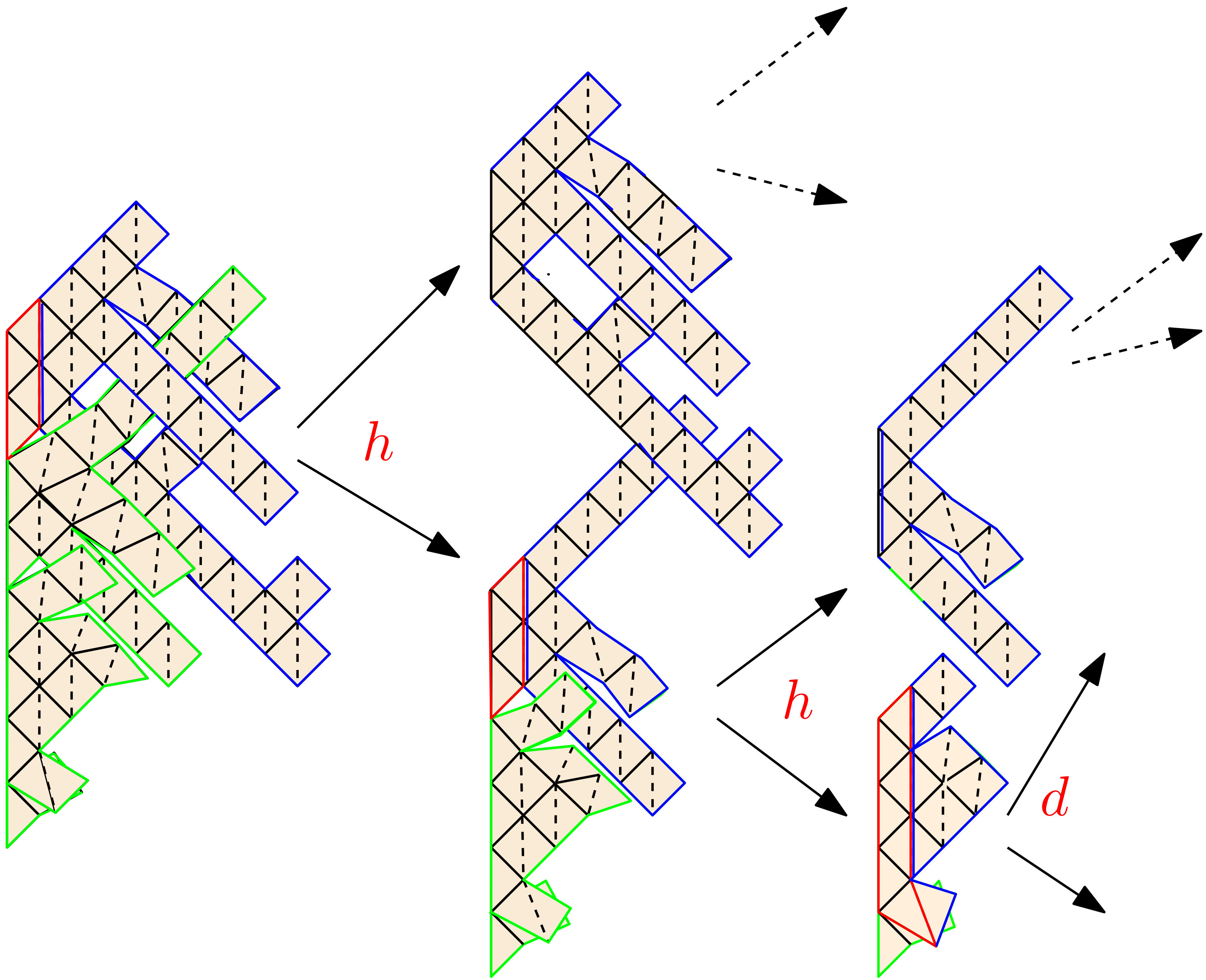

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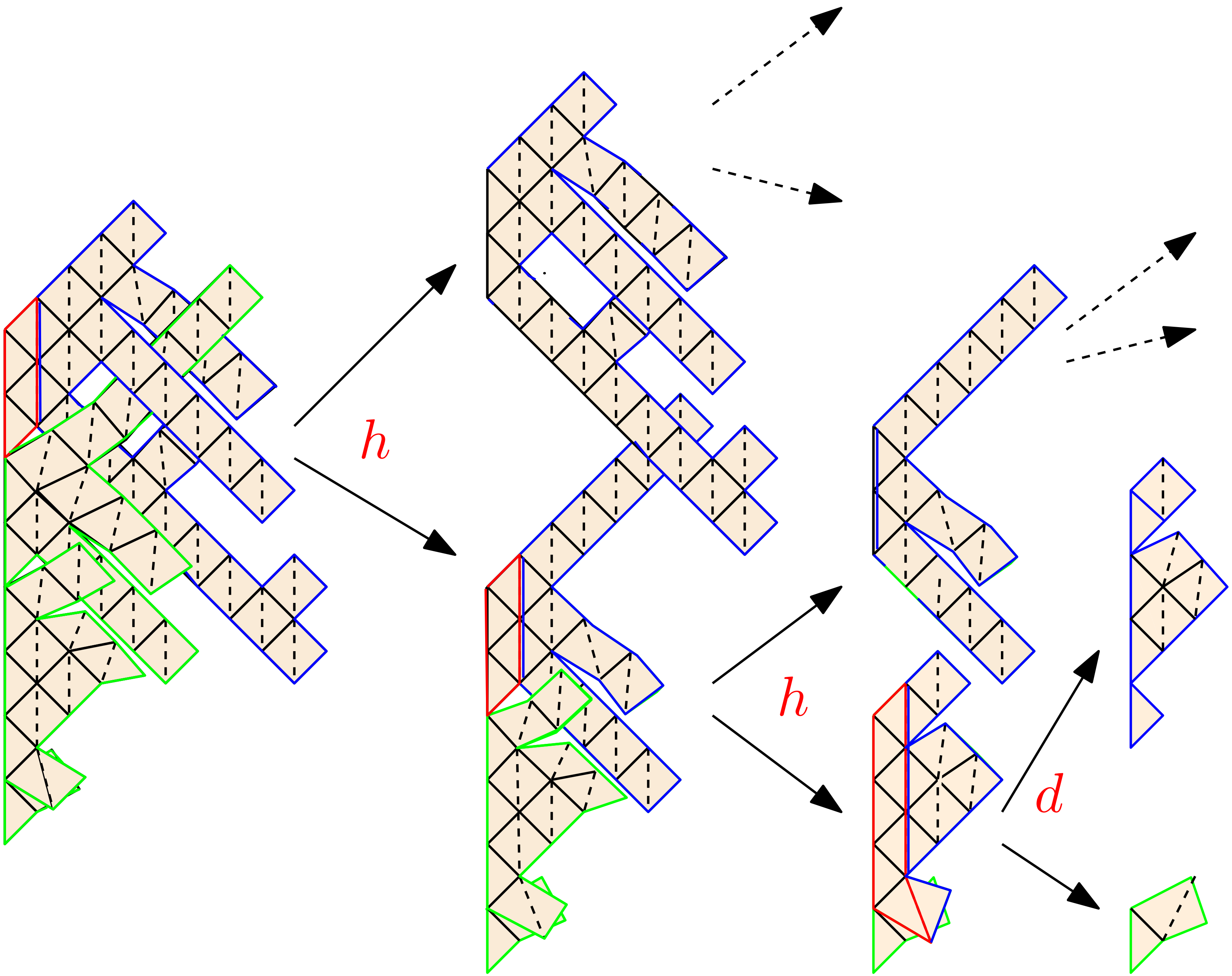

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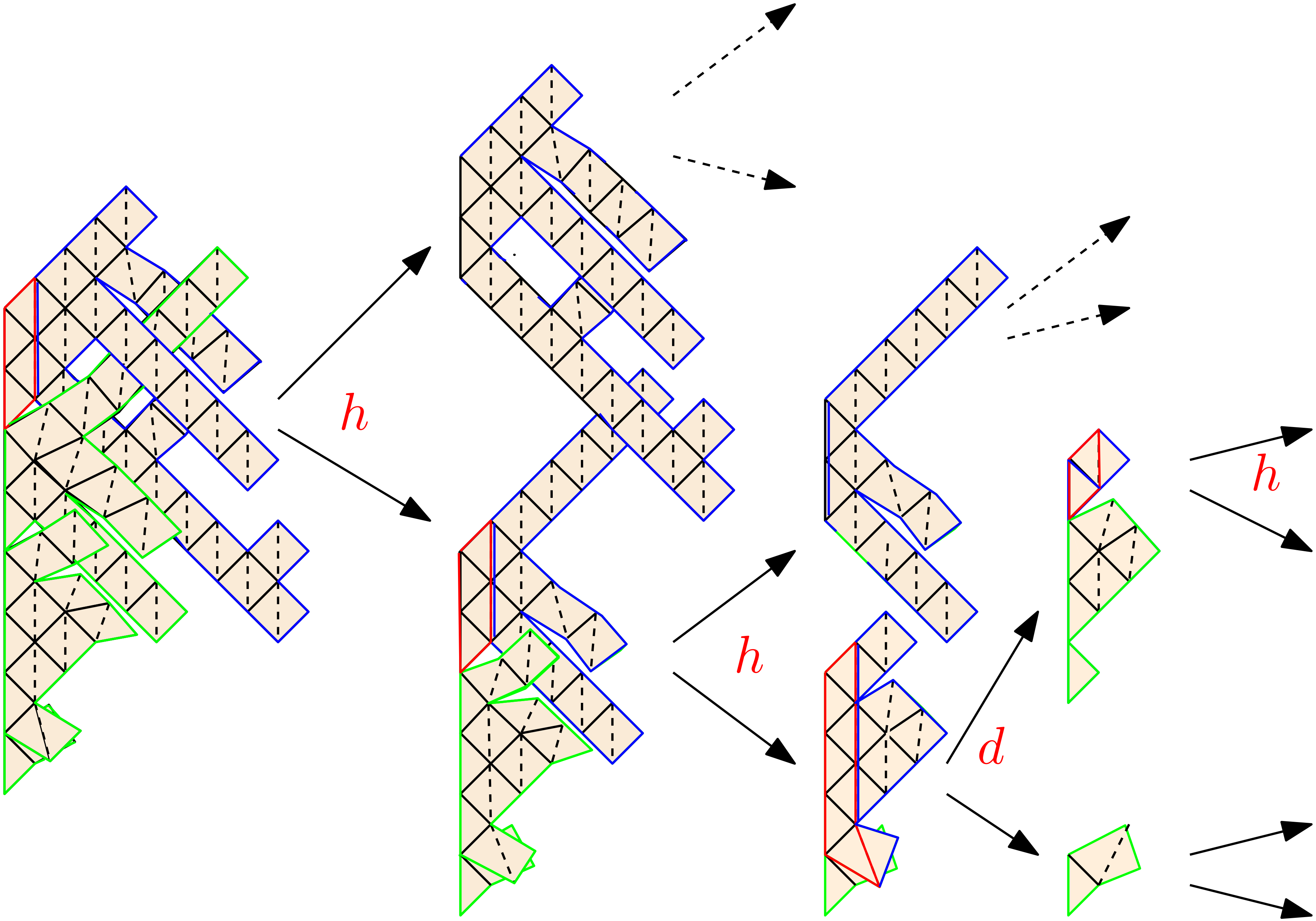

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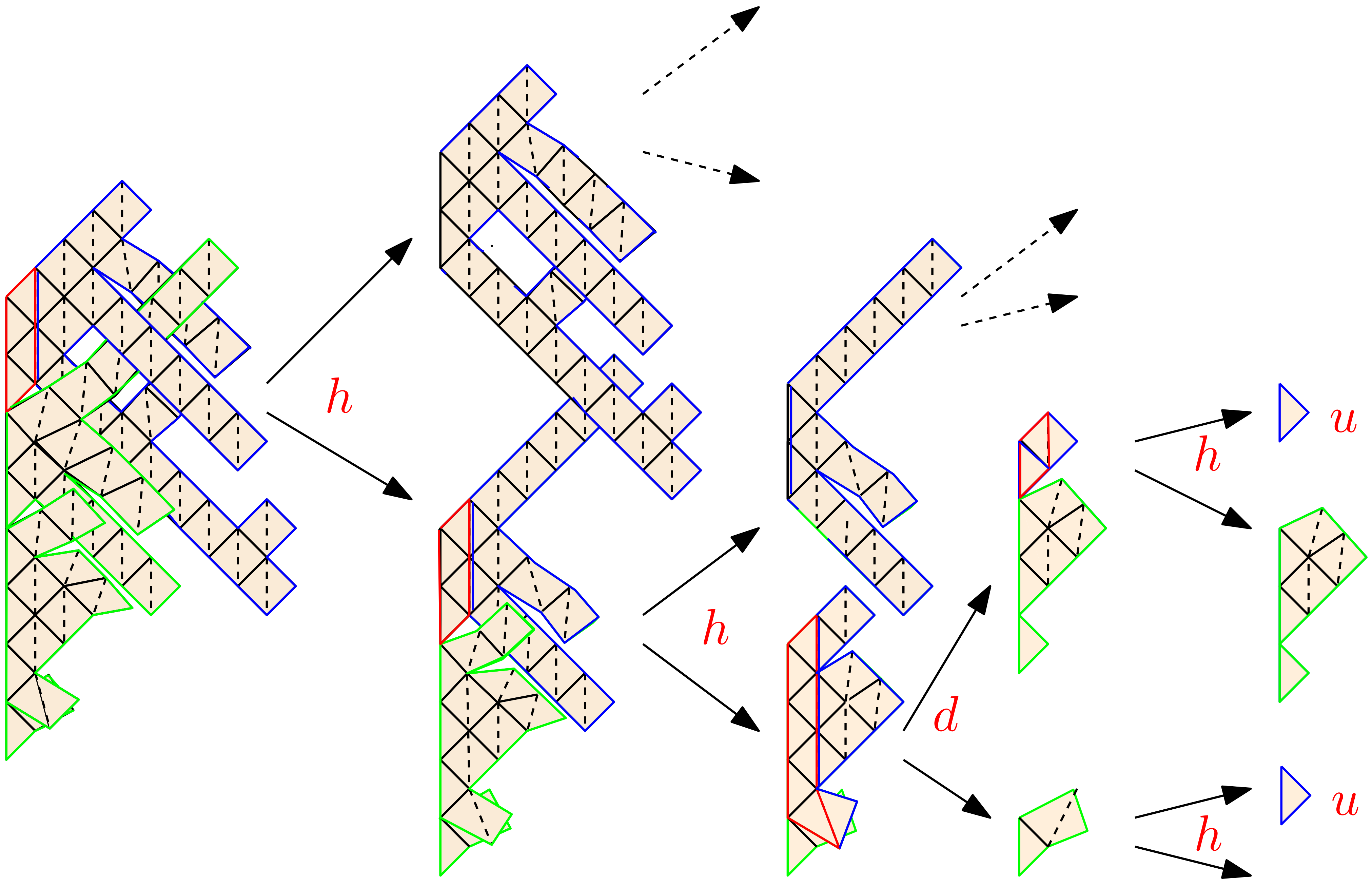

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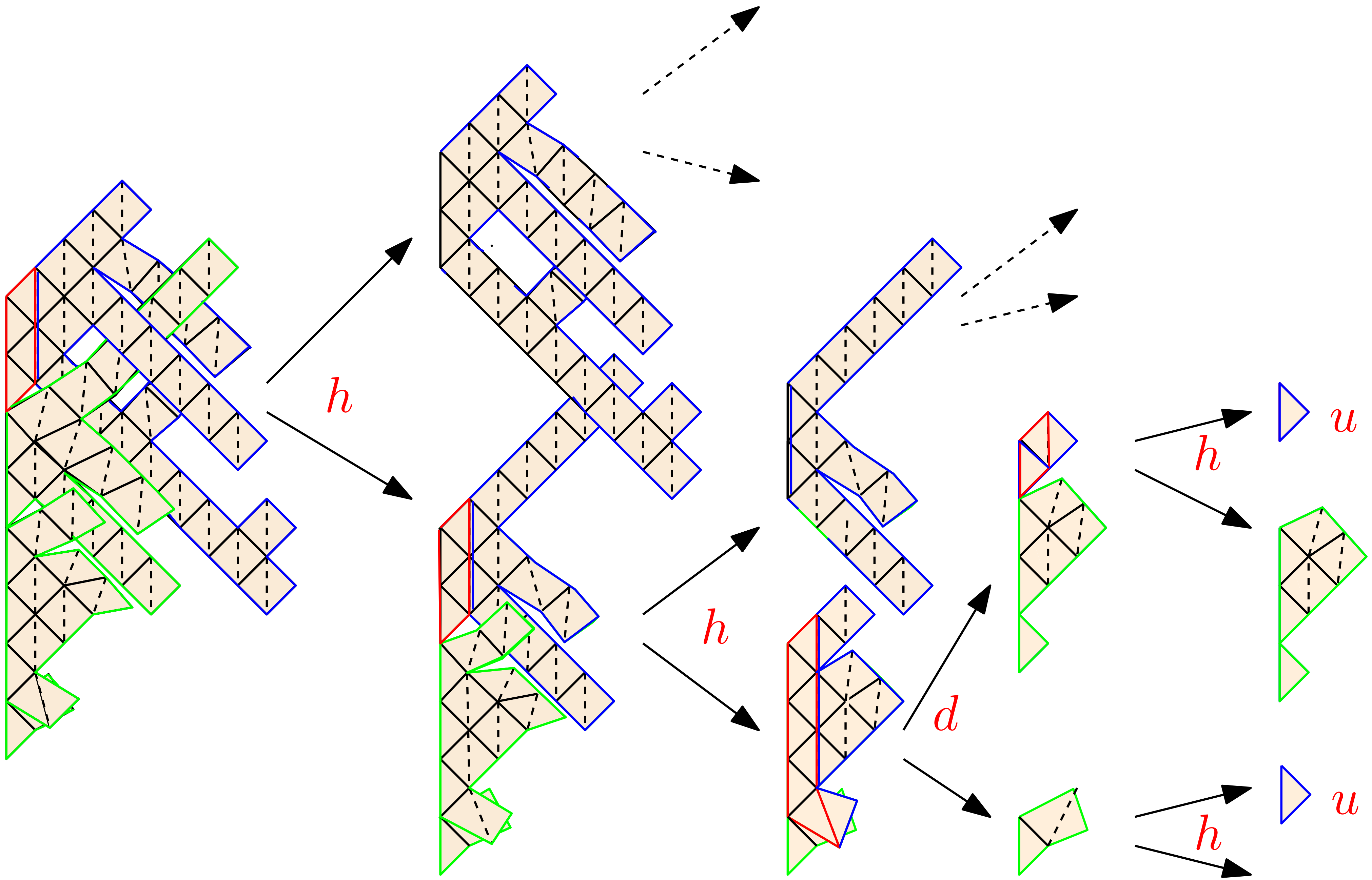

Fish tails vs fighting fish

Fish tails with height 1 and n free upper edges are in one-to-one correspondence with fighting fish with $n+1$ free upper edges.

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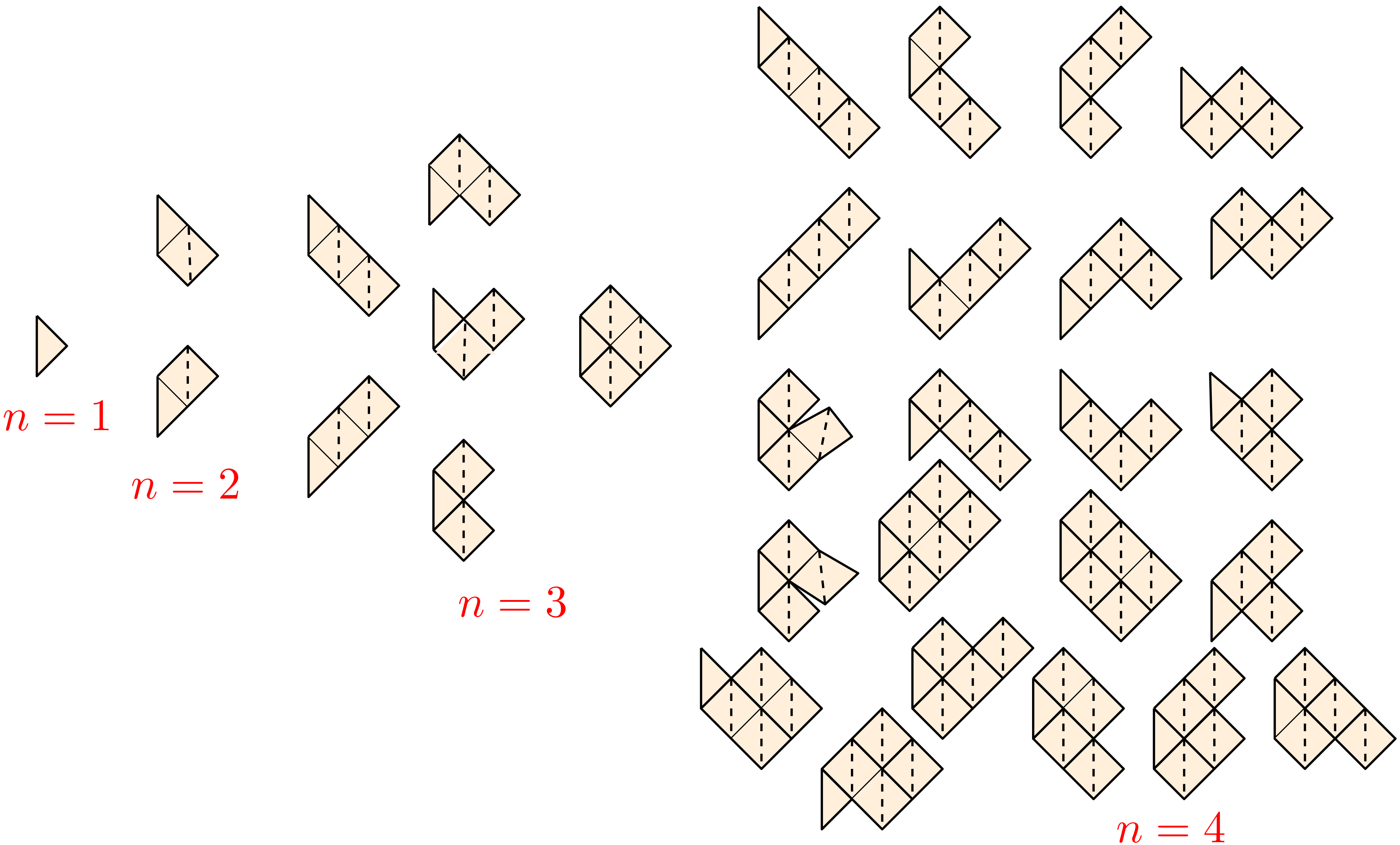

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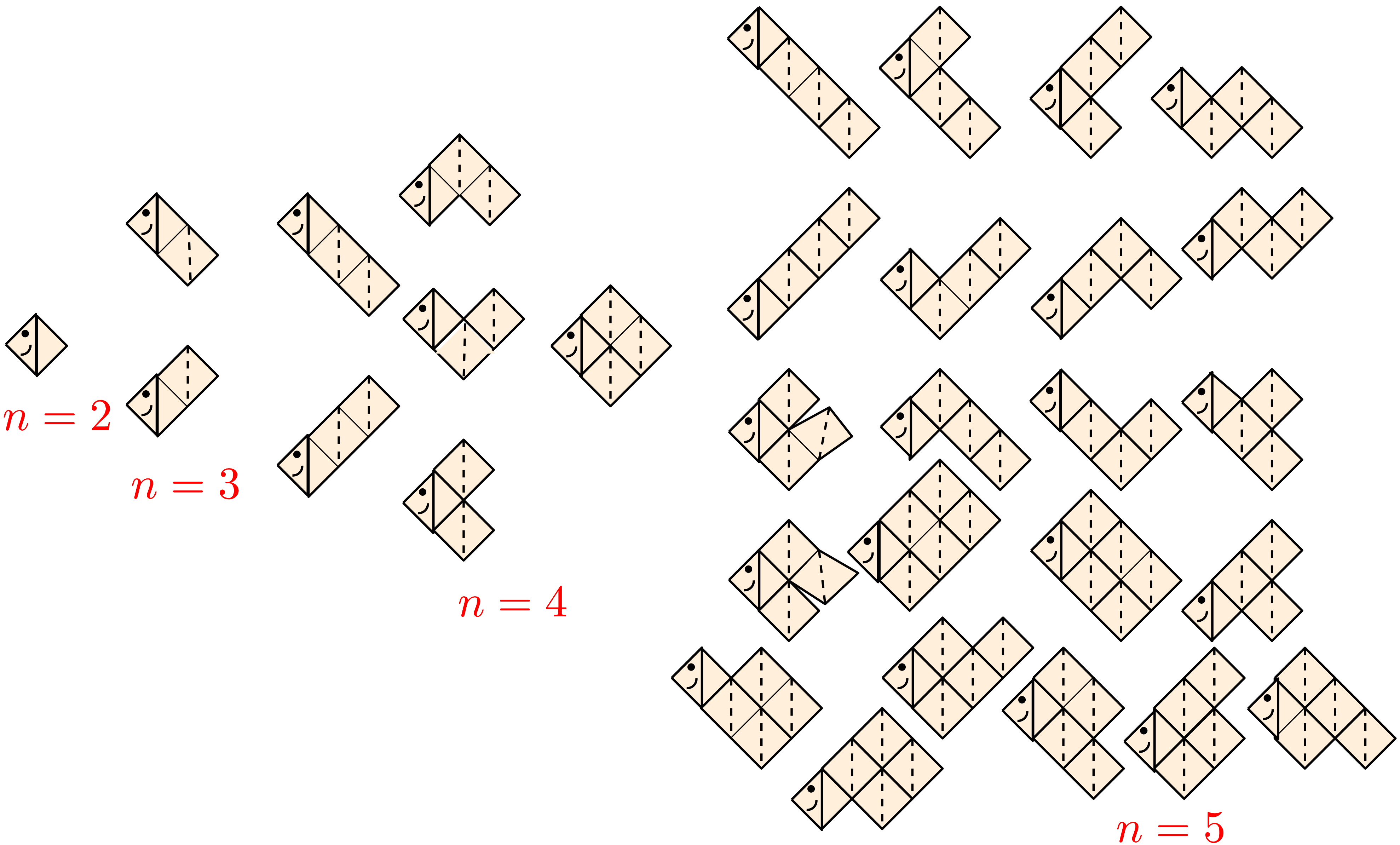

The generating function

Let $\mathcal{FT}$ be the set of fish tails without the empty fish tail.

Then $T(v, q, x, t) = \sum_{T \in \mathcal{FT}} v^{h(T)} q^{a(T)} x^{c(T)} t^{n(T)}$ denote the generating function of fish tails, where

$h(T)$ is the height of T

$a(T)$ is the area of T

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Let us denote

$$T(v,q) \equiv T(v,q,x,t) \qquad\qquad f(q) = [v]T(v,q)$$

We are going to write the functional equation associated with the previous construction.

The functional equation for fish tails

$T(v,q,x,t) = \sum_{T\in\mathcal{F}\mathcal{T}} v^{h(T)} q^{a(T)} x^{c(T)} t^{n(T)}$

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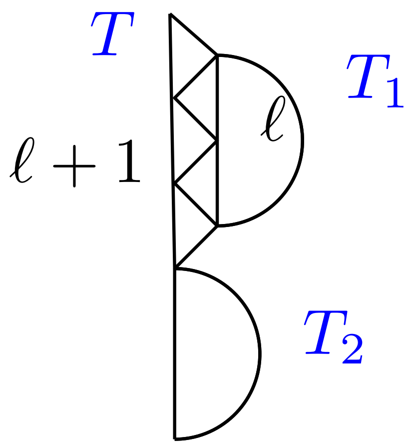

Operation u

$$n(T) = n(T_1) + n(T_2) + 1$$

$$h(T) = h(T_1) + h(T_2) + 1$$

$$c(T) = c(T_1) + c(T_2) \text{ if } \ell := h(T_1) \neq 0$$

$$c(T) = c(T_2) + 1 \quad \text{if } \ell = 0$$

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The functional equation for fish tails

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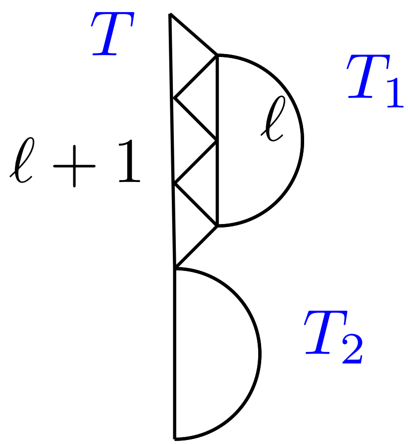

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Operation u gives the term

$$tvqT(vq^2,q,x,t)(T(v,q,x,t)+1) + tvqx(T(v,q,x,t)+1)$$

$$= tvq(T(vq^2,q)+x)(T(v,q)+1)$$

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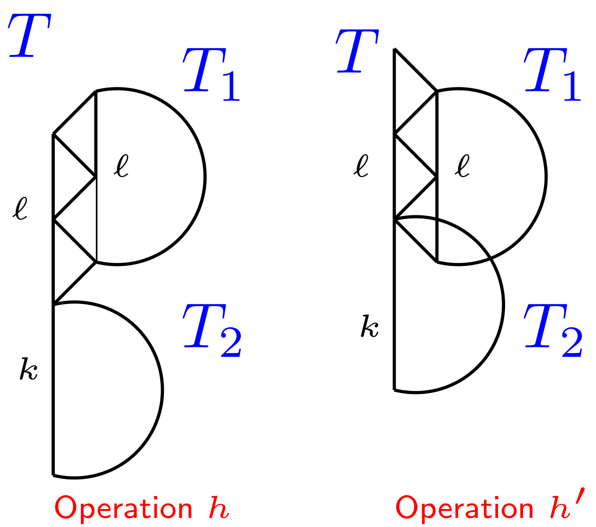


$$n(T) = n(T_1) + n(T_2) + 1$$

$$h(T) = h(T_1) + h(T_2)$$

$$c(T) = c(T_1) + c(T_2) \ \ (\ell \neq 0)$$

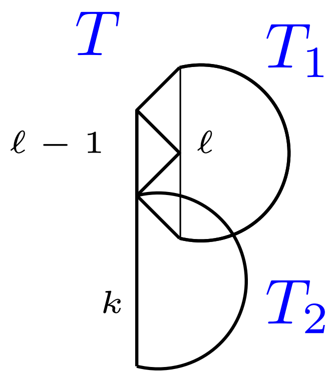
$$a(T) = a(T_1) + a(T_2) + 2\ell$$

Operation h and h' give the term

$$2t(T(vq^2, q)(T(v,q) + 1)$$

The functional equation for fish tails

$$T(v,q,x,t) = \sum_{T \in \mathcal{FT}} v^{h(T)} q^{a(T)} x^{c(T)} t^{n(T)}$$



Operation d

$$n(T) = n(T_1) + n(T_2) + 1$$

$$h(T) = h(T_1) + h(T_2) - 1 \qquad (\ell > 1)$$

$$c(T) = c(T_1) + c(T_2)$$

$$a(T) = a(T_1) + a(T_2) + 2\ell - 1$$

Operation u gives the term

$$\frac{t}{vq}(T(vq^2,q) - vq^2 f(q))(T(v,q)+1)$$

Enumeration wrt the perimeter and number of tails

The functional equation

$$\begin{aligned} T(v,q) = \quad & tvq(T(vq^2,q)+x)(T(v,q)+1) + 2tT(vq^2,q)(T(v,q)+1) \\ & + \tfrac{t}{vq}(T(vq^2,q) - vq^2 f(q))(T(v,q)+1) \end{aligned}$$

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Letting $q=1$ the master equation reduces to

$$T(v) = \quad tv(T(v)+x)(T(v)+1) + 2tT(v)(T(v)+1)$$

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where $T(v) \equiv T(v,1)$ and $f \equiv f(1)$

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This equation is now a polynomial equation with one catalytic variable and it admits an algebraic solution.
(Bousquet-Mélou and Jehanne, *J. Combin. Theory Ser.B*, 2006)

Enumeration wrt semi-perimeter and number of tails

$$\begin{aligned} T(v) = \quad & tv(T(v)+x)(T(v)+1) + 2tT(v)(T(v)+1) \\ & + \tfrac{t}{v}(T(v)-vf)(T(v)+1) \end{aligned}$$

We apply the Bousquet-Mélou Jehanne trick:

Enumeration wrt semi-perimeter and number of tails

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Upon deriving with respect to v we obtain the following equation

$$(1 - tv(T(v)+x) - 2tT(v) - \frac{t}{v}(T(v)-vf) - (tv+2t+\frac{t}{v})(T(v)+1))\frac{dT(v)}{dv} =$$
$$= (T(v)+1)(t(T(v)+x) - \frac{t}{v^2}(T(v)-vf) - \frac{t}{v}f)$$

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$$= (T(v)+1)(t(T(v)+x) - \tfrac{t}{v^2}(T(v)-vf) - \tfrac{t}{v}f)$$

There is a power series $V \equiv V(t)$, such that setting $v = V$ cancels the left hand side:

$$(1 - tV(T(V)+x) - 2tT(V) - \tfrac{t}{V}(T(V)-Vf) - (tV+2t+\tfrac{t}{V})(T(V)+1)) = 0$$

we then also have $$(T(V)+1)(t(T(V)+x) - \tfrac{t}{V^2}(T(V)-Vf) - \tfrac{t}{V}f) = 0$$

and the main equation gives a third equation.

Enumeration wrt semi-perimeter and number of tails

Simplifying the previous system of equations we obtain

$$\begin{aligned} V &= t\left(1 + V + \tfrac{xV^2}{1-V}\right)^2 \\ f &= xV - x^2 \tfrac{V^3}{(1-V)^2} \\ T(V) &= \tfrac{xV^2}{1-V^2} \end{aligned}$$

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The number of fighting fish with size $n+1$

$$
\begin{aligned}
V &= \tfrac{t}{(1-V)^2} \\
f &= V - \tfrac{V^3}{(1-V)^2}
\end{aligned}
$$

Lagrange inversion formula

$$\longrightarrow [t^n]f = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

Enumeration wrt semi-perimeter and number of tails.

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$$[x]f = [x^0]V = V_0$$

where $V_0 = t(1+V_0)^2$

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Parallelogram polyominoes of size $n+1$

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Parallelogram polyominoes of size $n+1$

More generally generating function for fighting fish with size $n+1$ and c tails is rational in the Catalan generating function.

However explicit expressions are not particularly simple

Enumeration of fighting fish wrt the area

We are going to count fighting fish weighted by their area.

Let $A \equiv A(x,t)$ be the total area generating function.

Then $A(x,t) = \sum_F a(F) t^{n(F)} = \frac{\partial(qf(q))}{\partial q}|_{q=1} = f + \frac{\partial(f(q))}{\partial q}|_{q=1}$

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By deriving with respect to q and by setting $q=1$ we obtain

$$(1 - tv(T(v,1)+x) - 2tT(v,1) - \tfrac{t}{v}(T(v,1) - vf) - (tv + 2t + \tfrac{t}{v})(T(v)+1))\frac{\partial T}{\partial q}(v,1)$$

$$= (T(v,1)+1)$$

$$\left((tv + 2t + \tfrac{t}{v}) \cdot 2v\frac{\partial T}{\partial v}(v,1) + tv(T(v,1)+x) - \tfrac{t}{v}(T(v,1) - vf) - 2tf - t\frac{\partial f}{\partial q}(1)\right)$$

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By setting $v = V$ we have

$$(tV + 2t + \tfrac{t}{V}) \cdot 2V \tfrac{\partial T}{\partial v}(V,1) + tV(T(V,1)+x) - \tfrac{t}{V}(T(V,1) - Vf) - 2tf - t\tfrac{\partial f}{\partial q}(1)$$

To obtain $\frac{\partial T}{\partial v}(V,1)$ we apply the kernel method to the second derivative of the master equation with respect to v.

The area generating function

The generating function $A \equiv A(x,t)$ for the total area of fighting fish with size $n+1$ satisfies

$$-V(1-V)^2A^2 + 2(1-V)^2(1-V^2+xV^2)A - 4xV(1-V^2+xV^2) = 0$$

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Extracting the coefficient of x, $[x]A = A_1$, yields

$$2(1-V_0)^2(1-V_0^2)A_1 - 4V_0(1-V_0^2) = 0$$

where $V_0 = [x^0]V$ is a Catalan generating function satisfying $V_0 = t(1+V_0)^2$
we obtain the generating function for the total area of parallelogram polyominoes

$A_1 = \frac{2V_0}{(1-V_0)^2} = \frac{2t}{1-4t}$ The simplification to a rational function of t is a well-known feature of parallelogram polyominoes.

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The average area for parallelogram polyominoes of size n is $4^n/C_n$
where C_n are the Catalan numbers.

The average area for parallelogram polyominoes of size n scales like $n^{\frac{3}{2}}$

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The generating function of the total area of fighting fish with c tails and size $n+1$ is a rational function of the Catalan generating function V_0

The area generating function

The average area A_n of fighting fish with size $n+1$ grows like $n^{\frac{5}{4}}$

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Indeed, let us reconsider the equation for V, f, and A,

$$V = \frac{t}{(1-V)^2}$$

and get asymptotics by singularity analysis

$$f = V - \frac{V^3}{(1-V)^2}$$

$$A = \frac{1}{V} - \sqrt{\frac{(1+V)(1-3V)}{V(1-V)}}$$

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We obtain:

$$[t^n]A \underset{n\to\infty}{\sim} cte \cdot n^{-\frac{5}{4}} t_c^{-n}$$

$$[t^n]f \underset{n\to\infty}{\sim} cte \cdot n^{-\frac{5}{2}} t_c^{-n}$$

Then the average area is

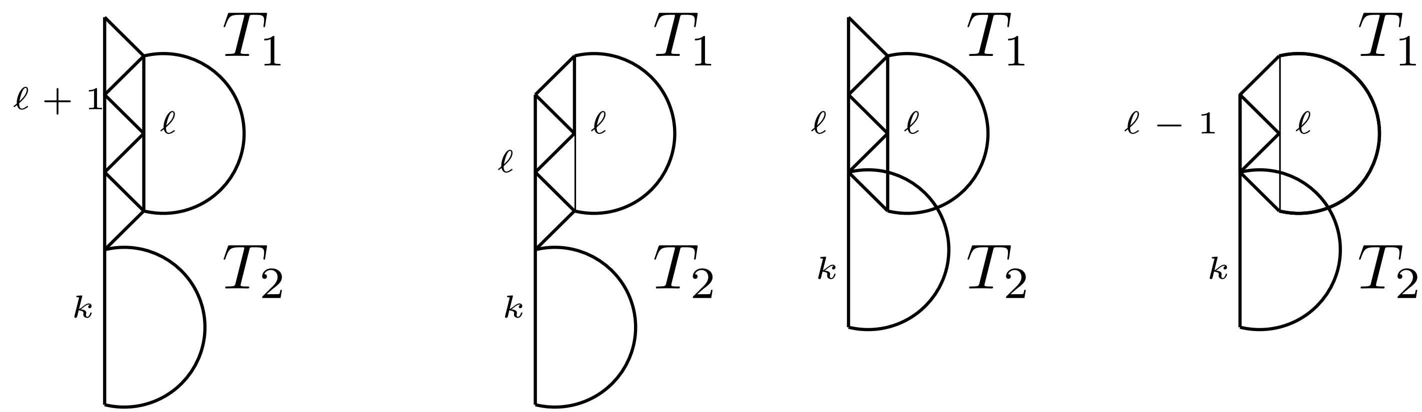
$$\frac{[t^n]A}{[t^n]f} \underset{n\to\infty}{\sim} cte \cdot n^{\frac{5}{4}}$$

A refinement of the main formula

$T(v,q,x,t)=\sum_{T\in\mathcal{FT}} a^{l(T)}b^{r(T)}v^{h(T)}q^{a(T)}x^{c(T)}t^{n(T)}$ denote the generating function of fish tails, where

$l(T)$ is the number of upper-left free edges of T

$r(T)$ is the number of upper-right edges of T



Operation u Operation h Operation h' Operation d

$$T(v,q)=\quad tvqb(T(vq^2,q)+x)(T(v,q)+1)+aT(vq^2,q)(T(v,q)+1)+$$

$$+bT(vq^2,q)(T(v,q)+1)+\frac{t}{vq}a(T(vq^2,q)-vq^2f(q))(T(v,q)+1)$$

$$\#\left\{\begin{array}{c}\text{fighting fish with}\\ i\text{ top left and }j\text{ top right edges}\end{array}\right\}=\frac{1}{(2i+j-1)(2j+i-1)}\binom{2i+j-1}{i}\binom{2j+i-1}{j}$$

Bijections and parameter equidistributions?

Sloane's Online Encyclopedia of Integer Sequences

$$\#\left\{\begin{array}{c}\text{fighting fish}\\ \text{with semi-perimeter } n+1\end{array}\right\} = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

1, 2, 6, 91, 408, 1938...

This integer sequence was already in Sloane's OEIS!

The number of fighting fish of size $n+1$ (with i left and j down top edges) is equal to the number of:

- Two-stack sortable permutations of $\{1,\ldots,n\}$ (i ascending and j descending runs) (West, Zeilberger, Bona, 90's)

- Rooted non separable planar maps with n edges ($i+1$ vertices, $j+1$ faces) (Tutte, Mullin and Schellenberg, 60's)

- Left ternary trees with n noeuds ($i+1$ even, j odd vertices) (Del Lungo, Del Ristoro, Penaud, 1999)

Left ternary trees and further equidistributions

Natural embedding of a ternary tree:

- root vertex has label 0
- vertex with label $i \Rightarrow$ left child $i+1$,
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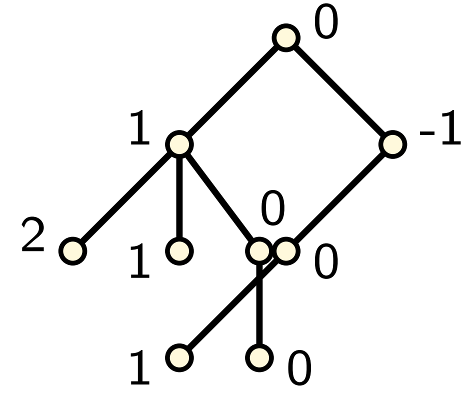
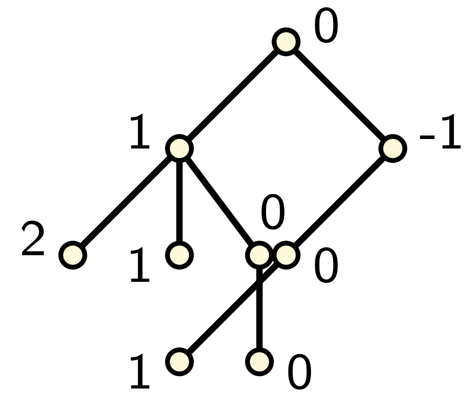


Left ternary trees and further equidistributions



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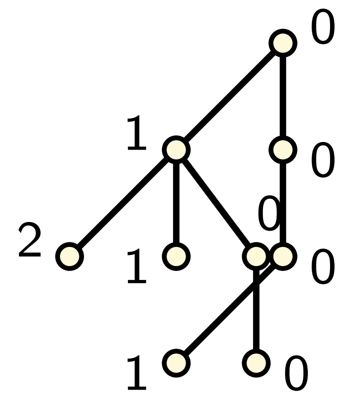

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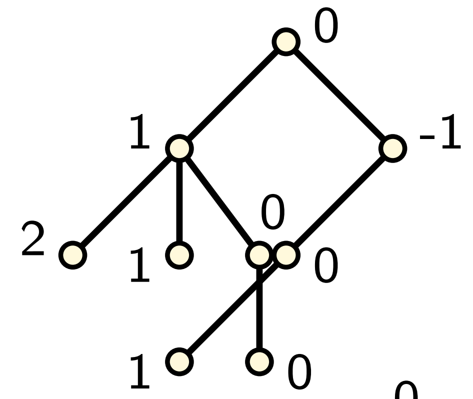

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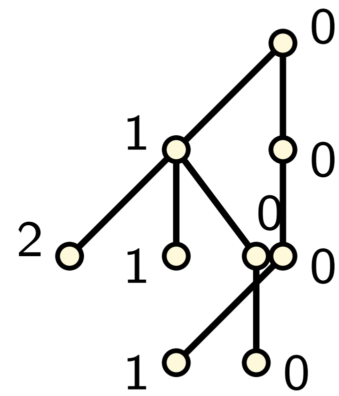

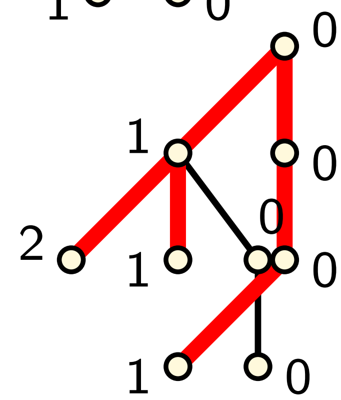

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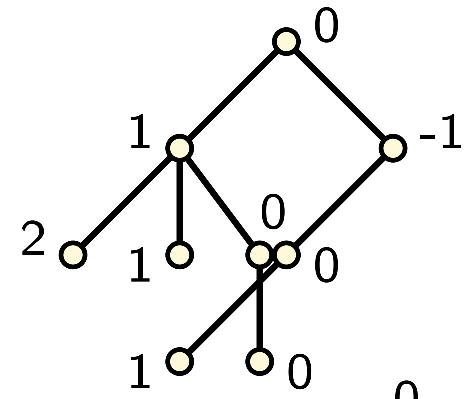

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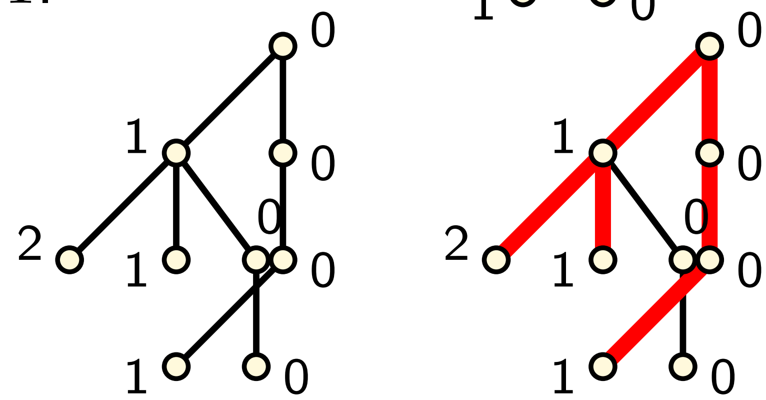

Theorem (DGRS 2016): The number of fighting fish with size $n+1$ and fin length k equals the number of left ternary trees with n nodes and core size k.

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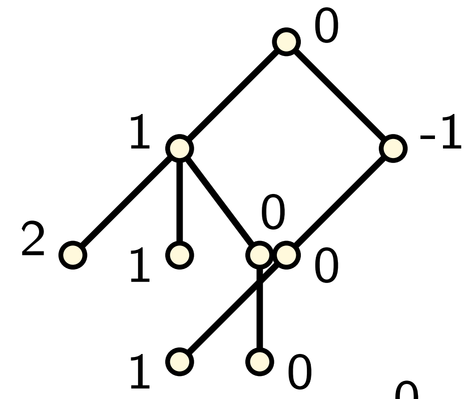

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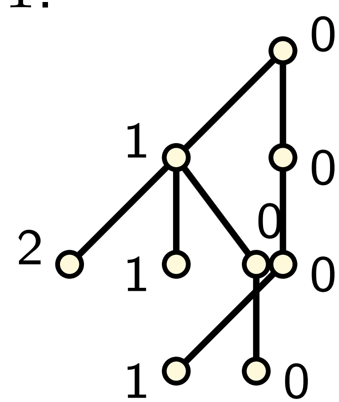

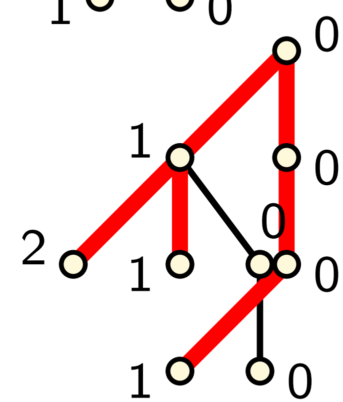

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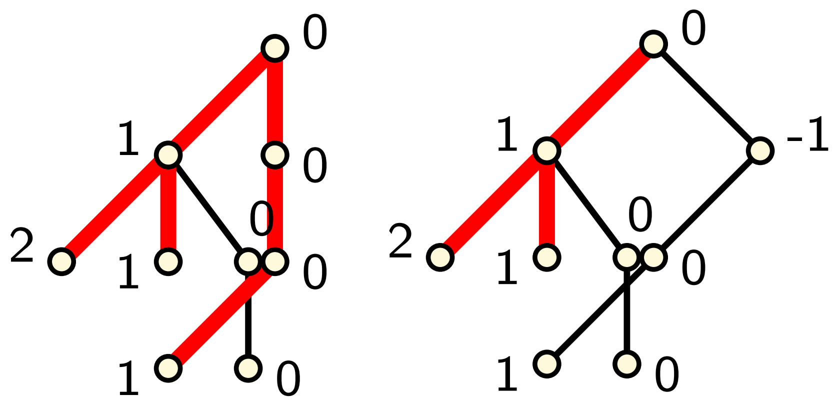
Proof? We computed the gf of fighting fish wrt size and fin length. Compute the gf of left ternary trees wrt size and core size...

Left ternary trees and further equidistributions



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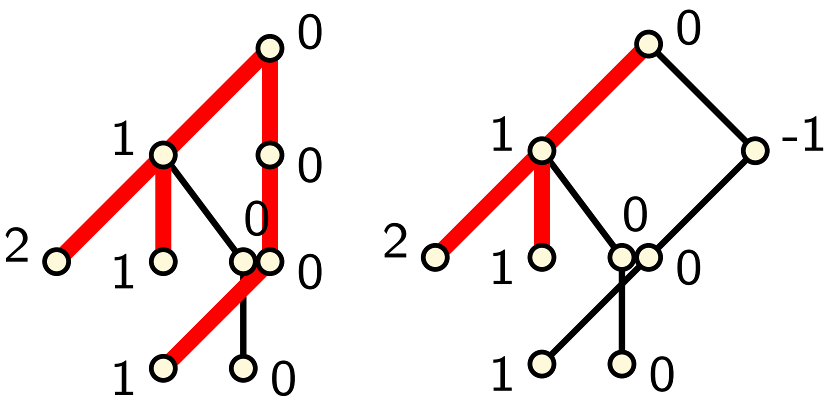

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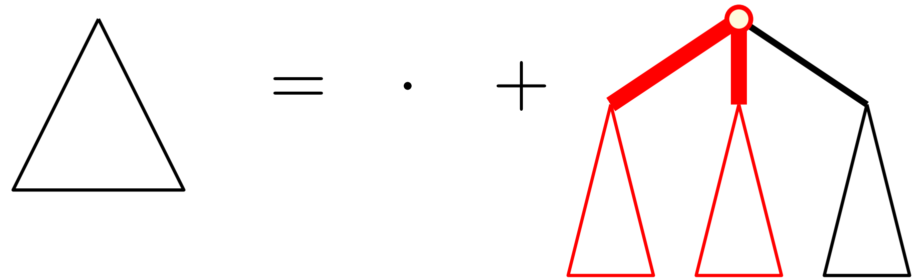

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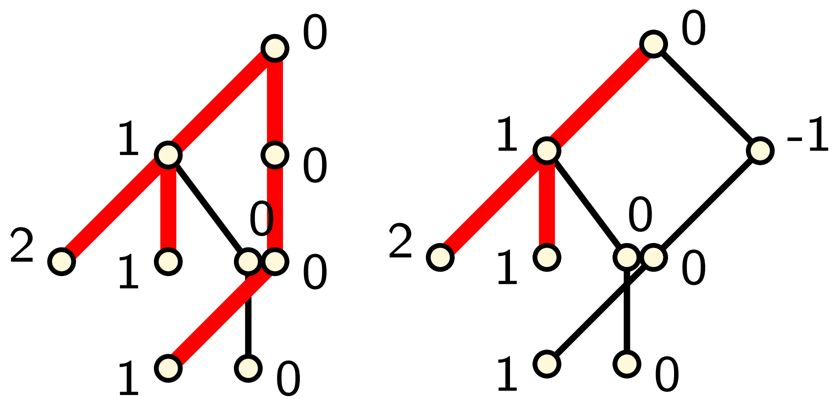

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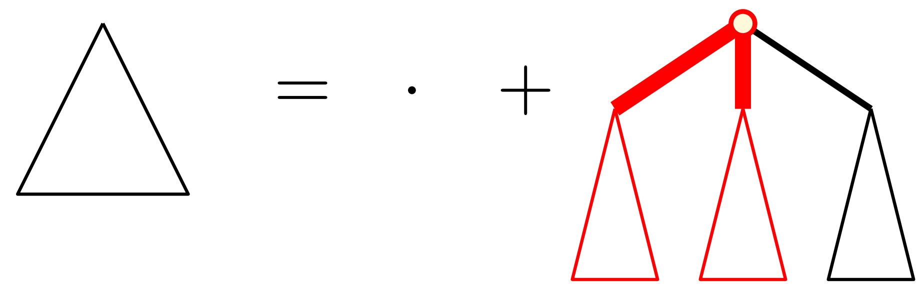

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Proposition The GF $\tau(u)$ of ternary trees wrt size and core size is

$$\tau(u) = 1 + tu\tau(u)^2\tau(1) \quad \text{generalizing} \quad \tau(1) = 1 + t\tau(1)^3$$

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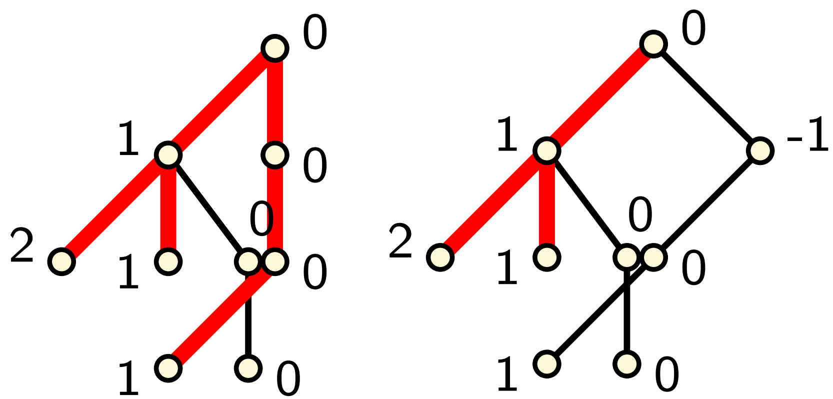

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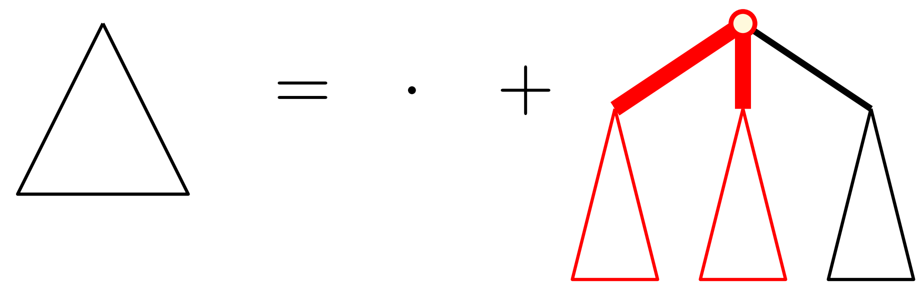

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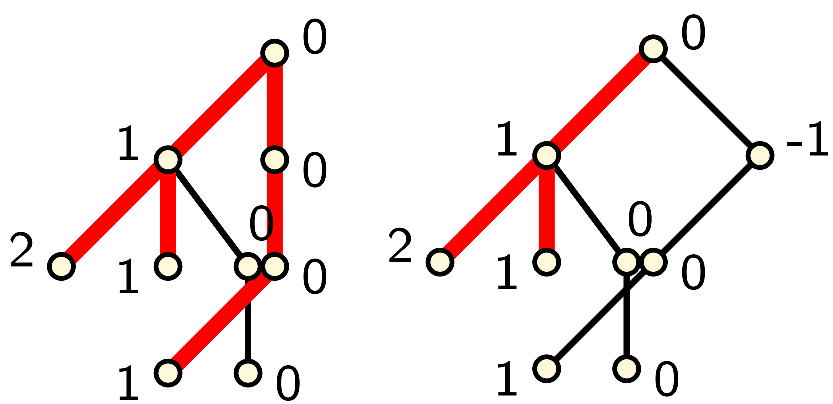

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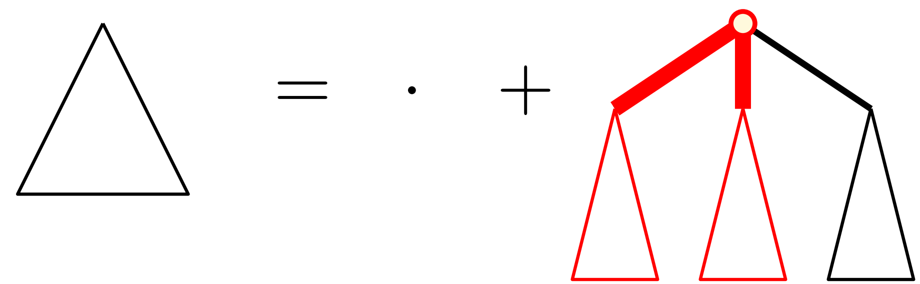

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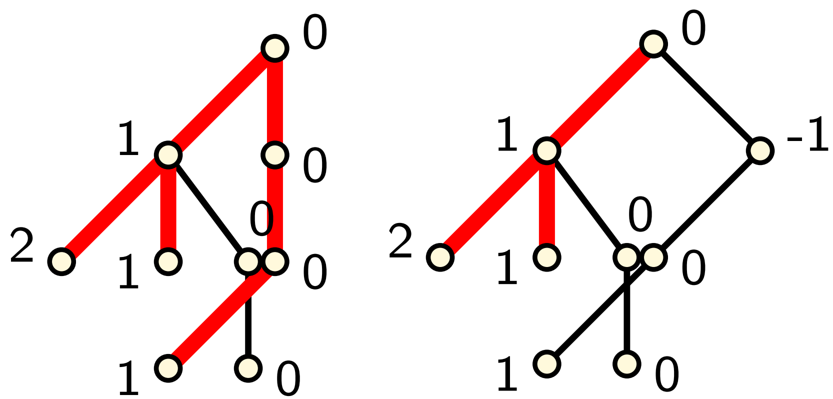

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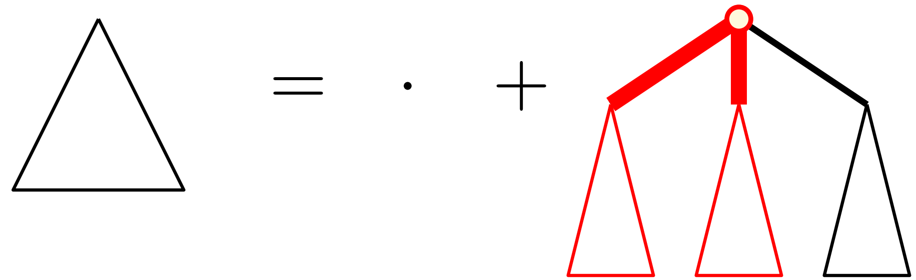

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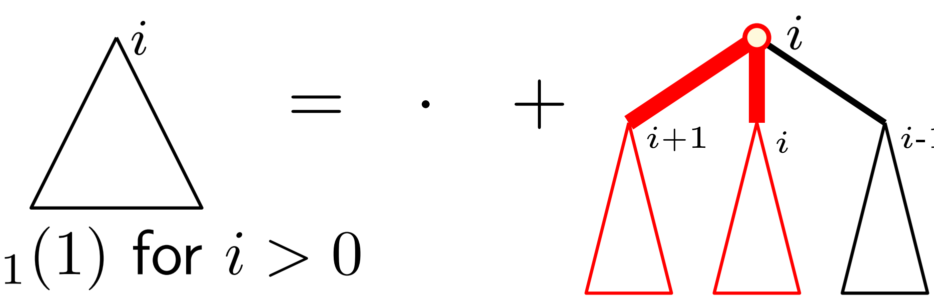
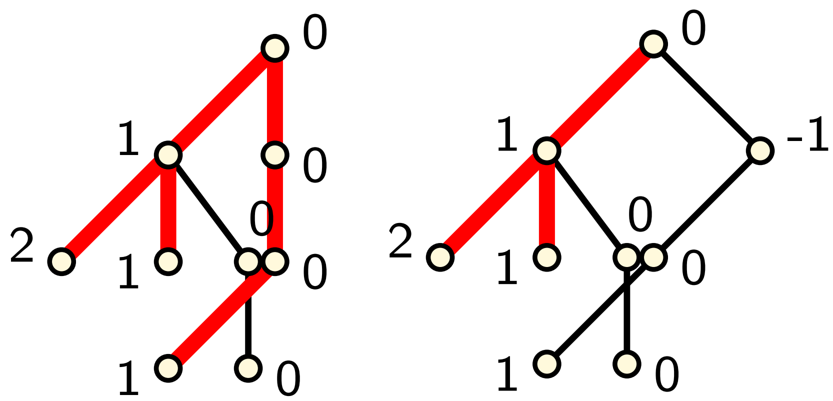
Proposition The GFs wrt size and core size of left ternary trees with root label i satisfy

$$\tau_0(u) = 1 + tu\tau_1(u)\tau_0(u)$$



$$\tau_i(u) = 1 + tu\tau_{i+1}(u)\tau_i(u)\tau_{i-1}(1) \text{ for } i > 0$$

Left ternary trees and further equidistributions



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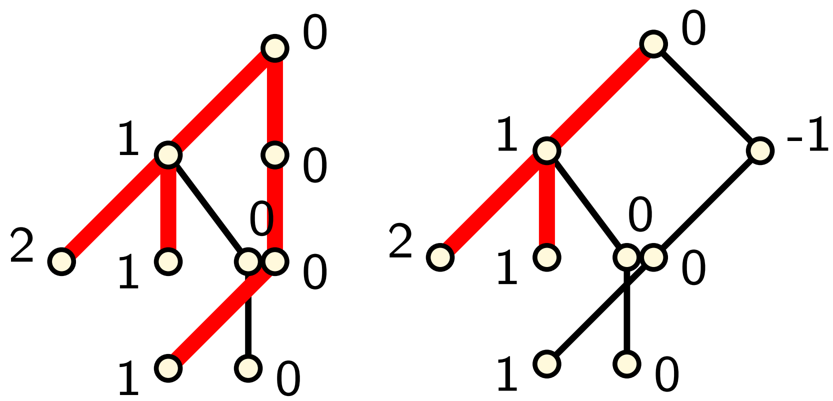
Left ternary trees and further equidistributions



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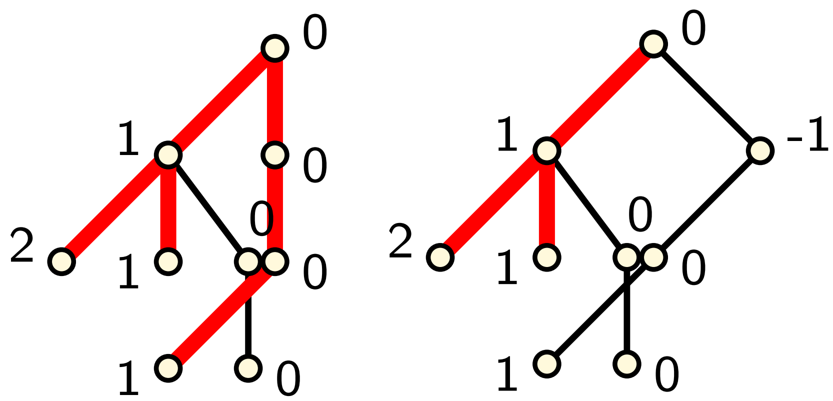

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Proof by guessing the formula and checking it satisfies the recurrence.

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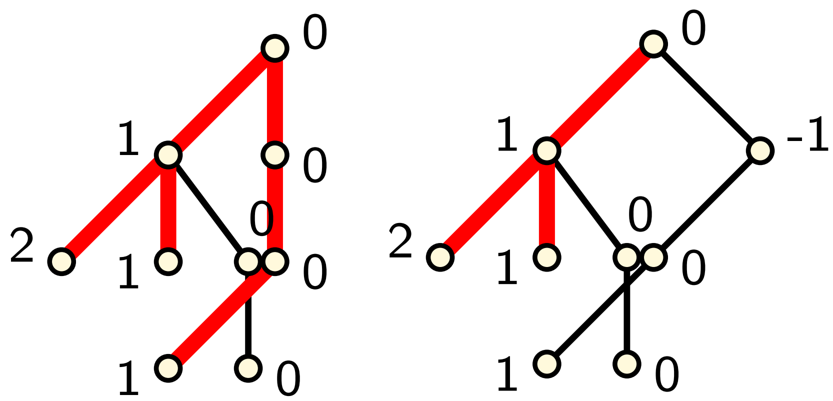

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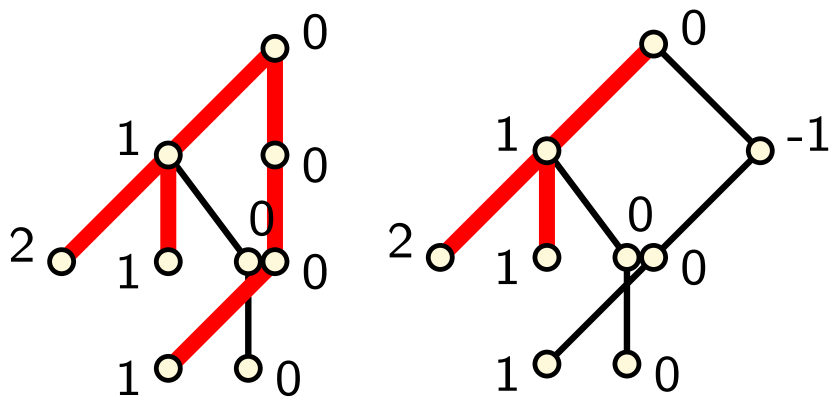

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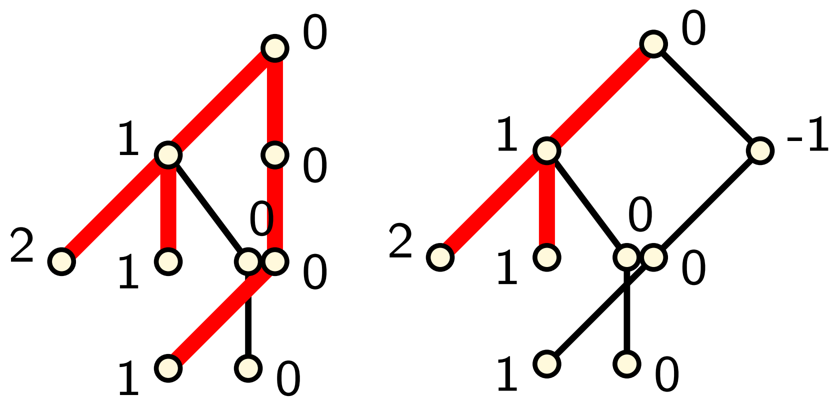
We extend the theorem and its proof by guessing the bivariate formula

Theorem (DGRS16) The bivariate size and core size GF of left ternary trees with label i is

$$\tau_i(u) = \tau(u) \frac{H_i(u)}{H_{i-1}(u)} \frac{1-X^{i+2}}{1-X^{i+3}} \quad \text{where} \begin{cases} \tau(u) = 1 + tu\tau\tau(u)^2 \\ H_j(u) = (1 - X^{j+1})XT(u) \\ \qquad -(1+X)(1 - X^{j+2}) \end{cases}.$$

Left ternary trees and further equidistributions



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Conjecture (DGRS 2016): The previous computation can be refined to prove joined equidistribution of:

fin length $\leftrightarrow$ core size
number of tails $\leftrightarrow$ number of right branches
number of left/right free edges $\leftrightarrow$ number of even/odd labels

THANK YOU

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fighting fish

2SS-permutations

left ternary trees

ns planar maps

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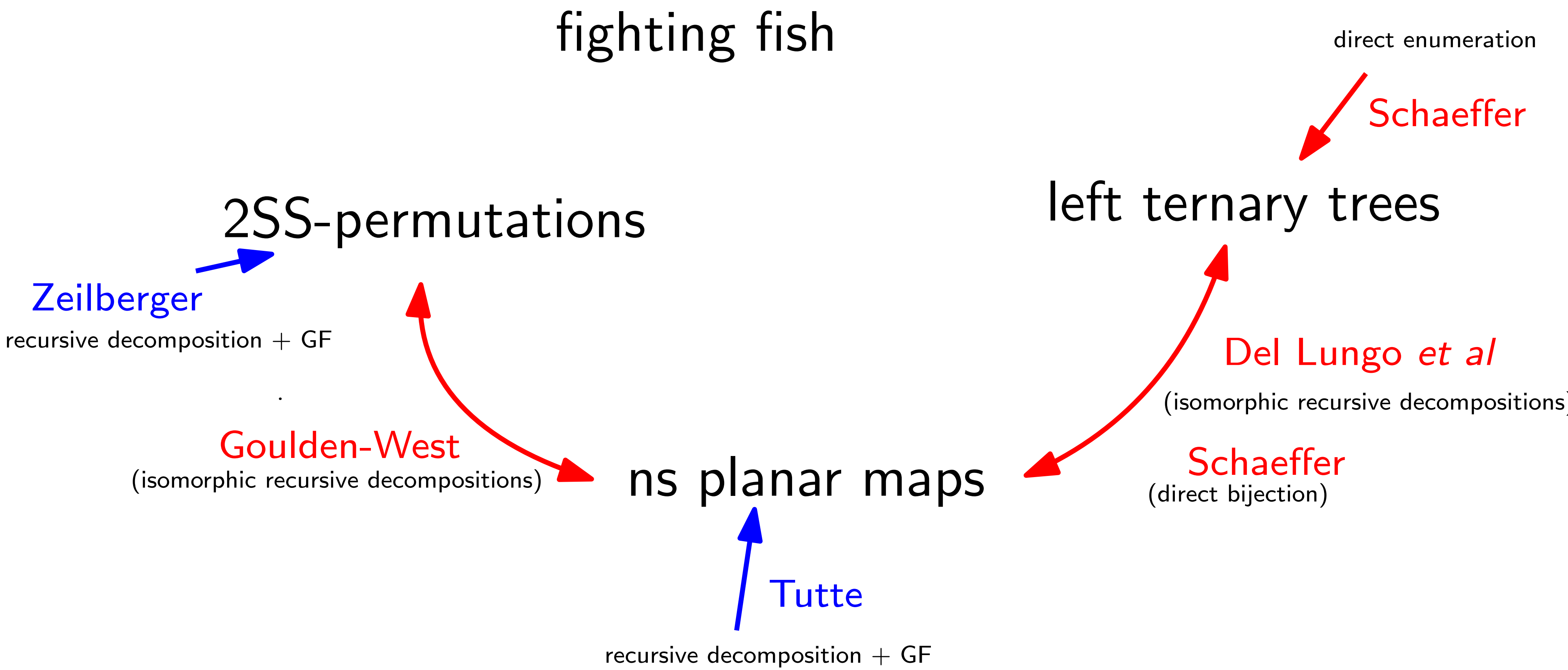
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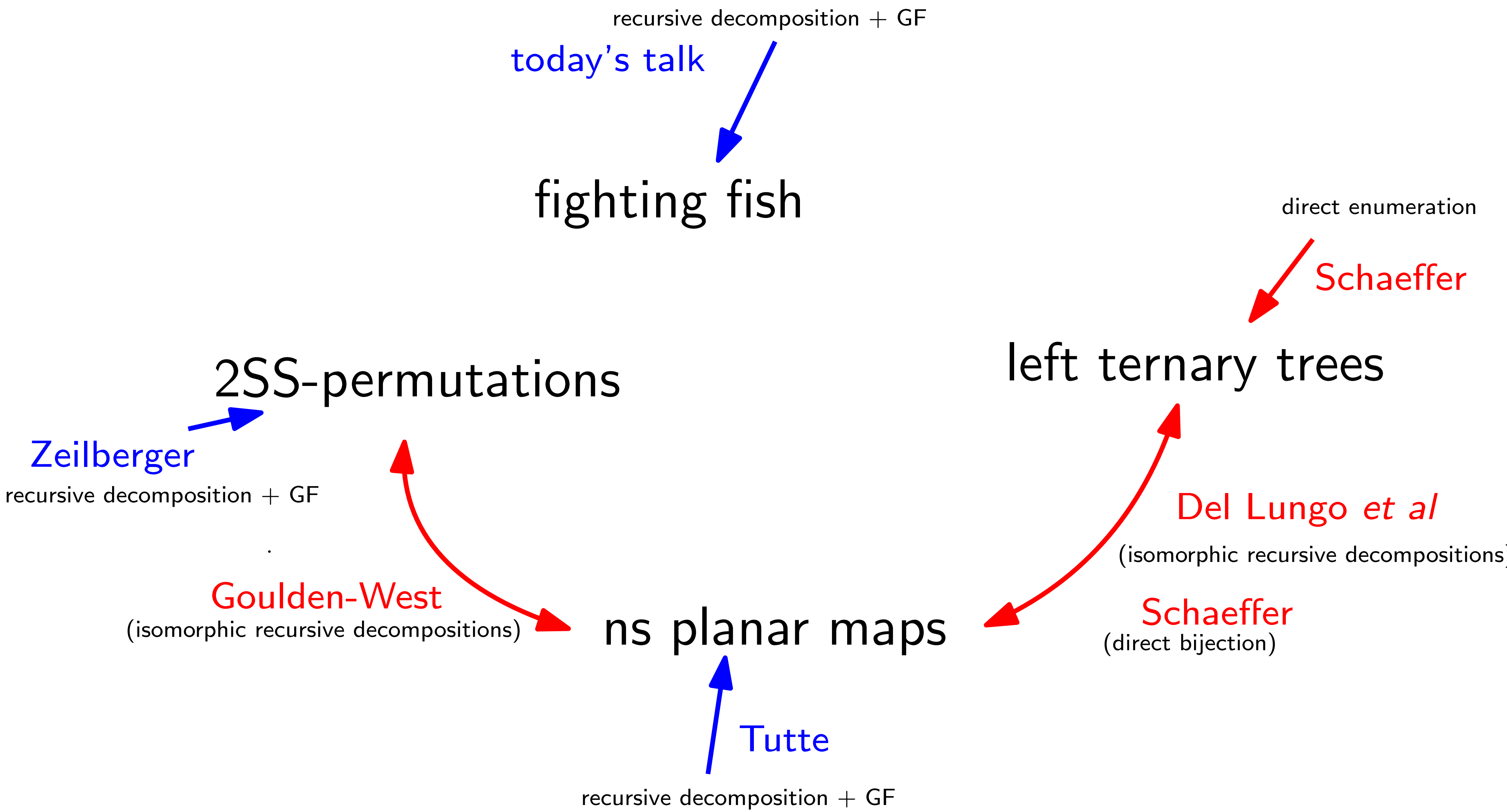

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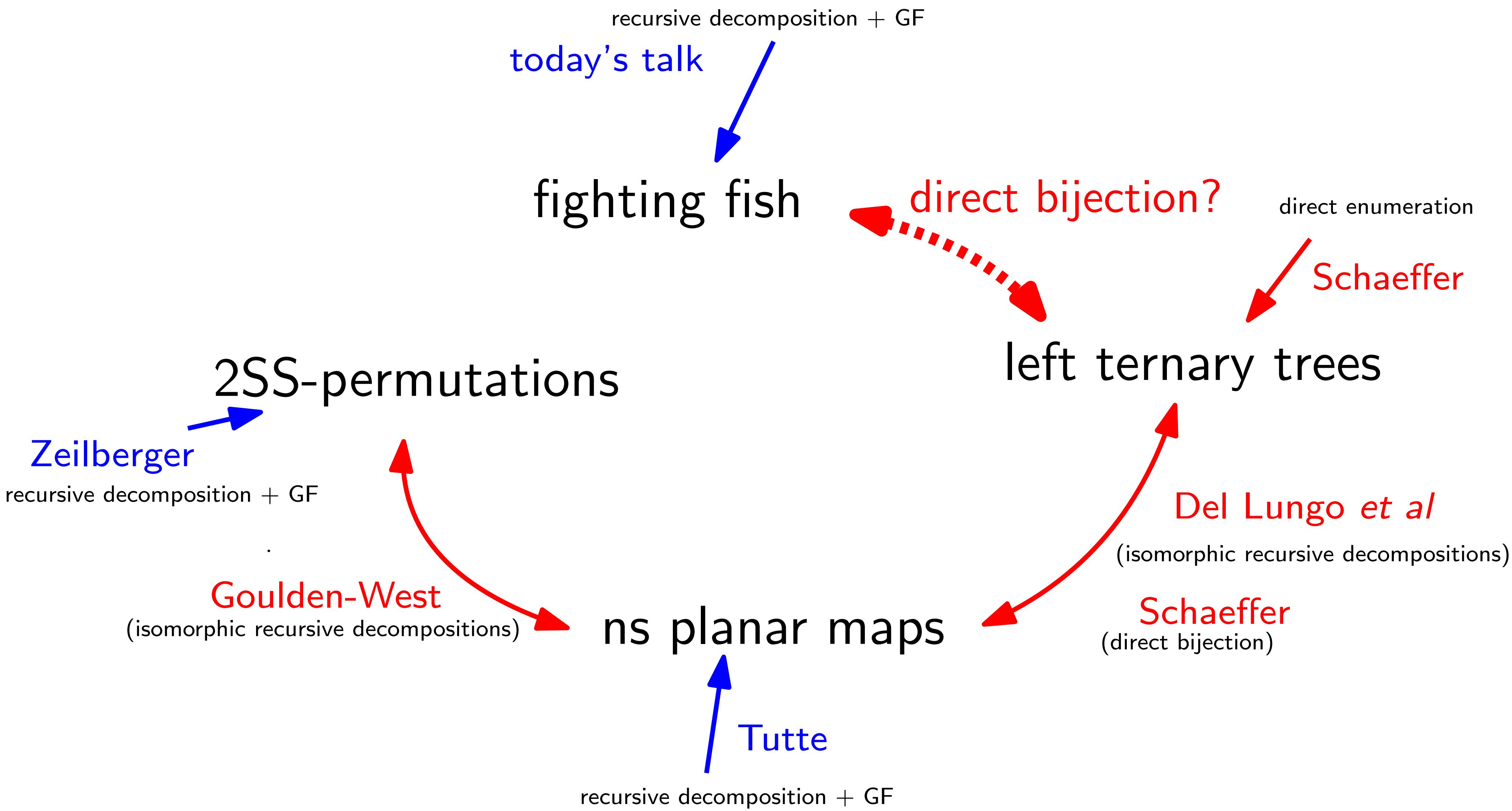
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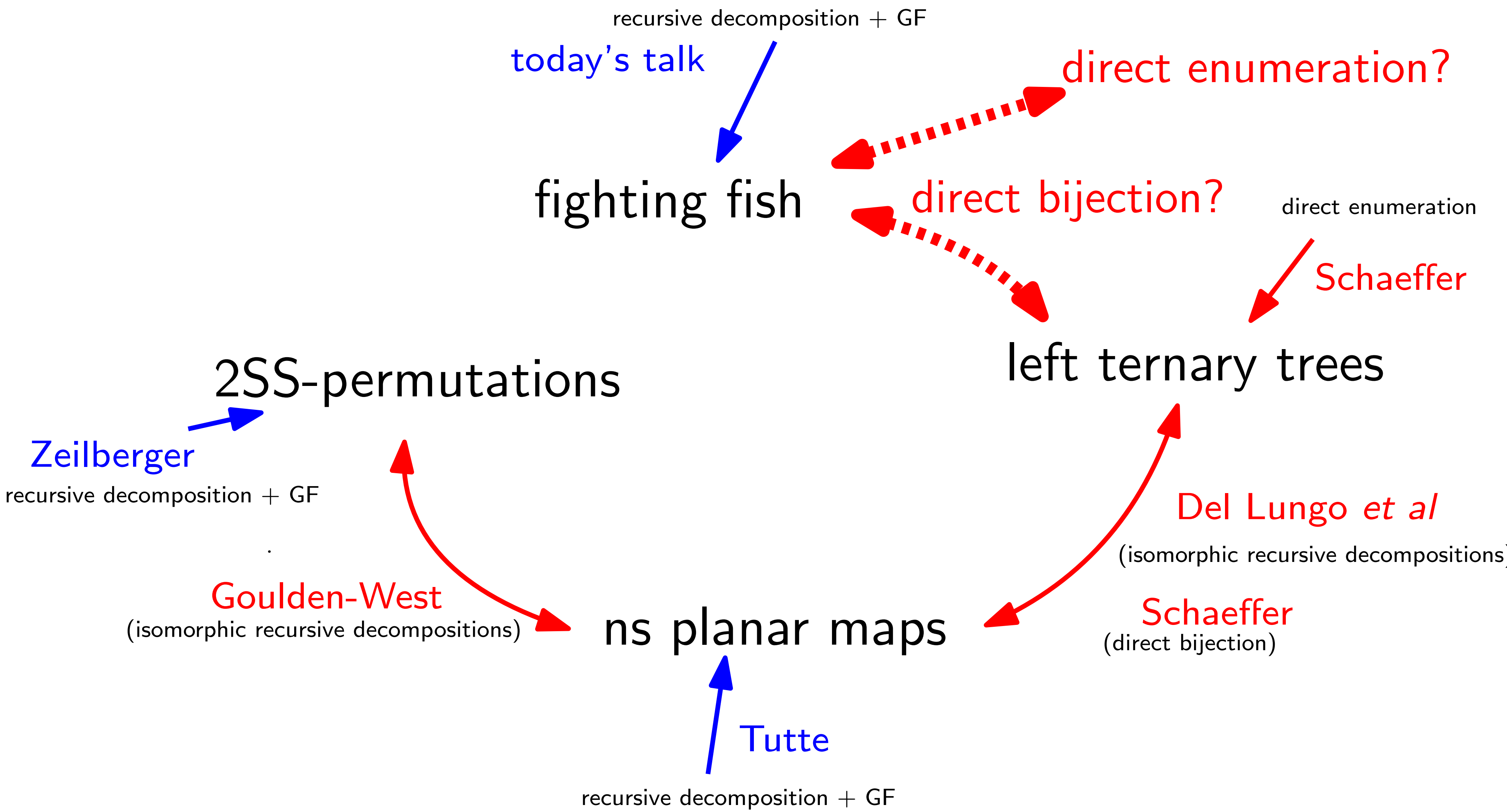
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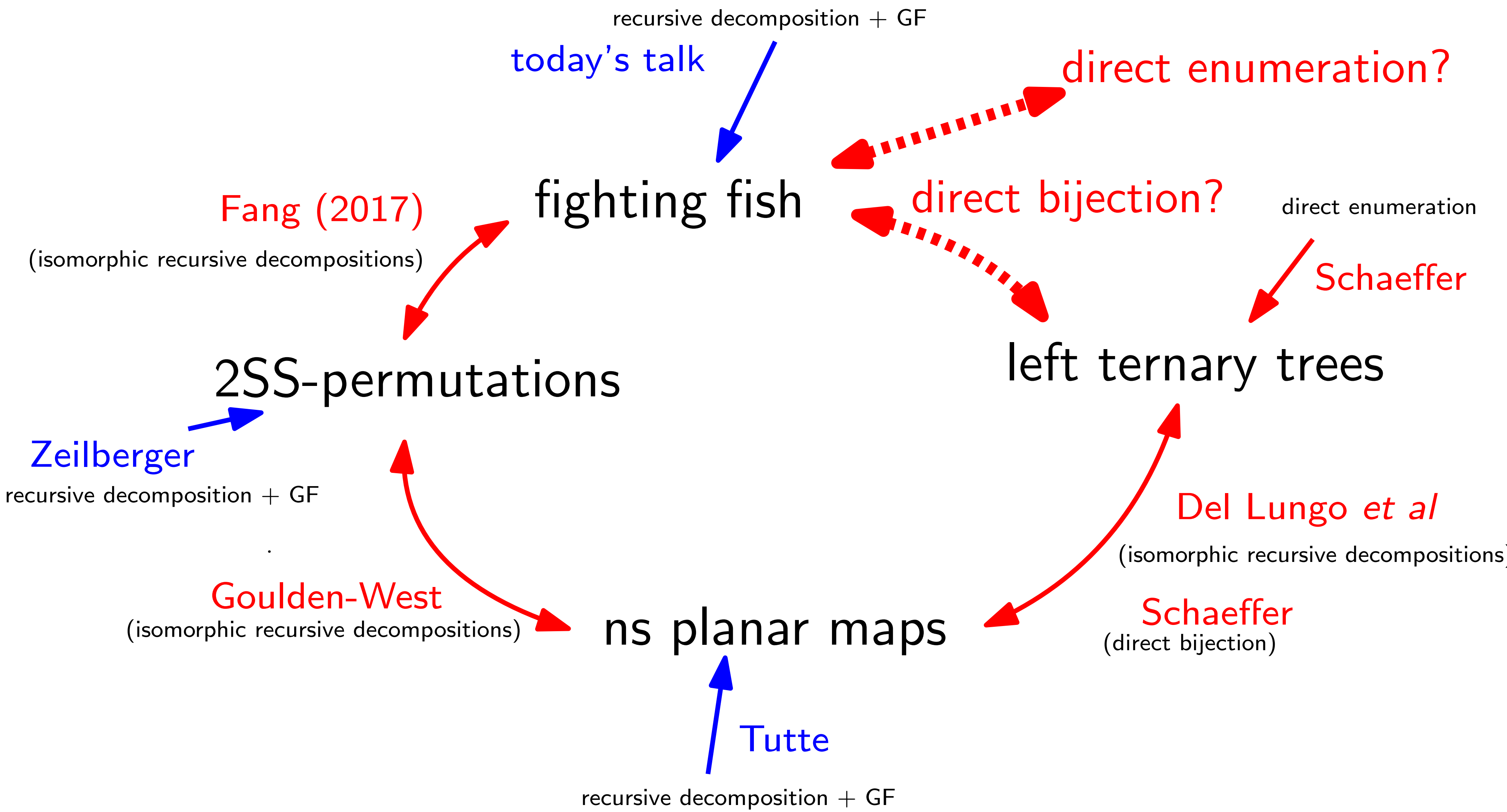

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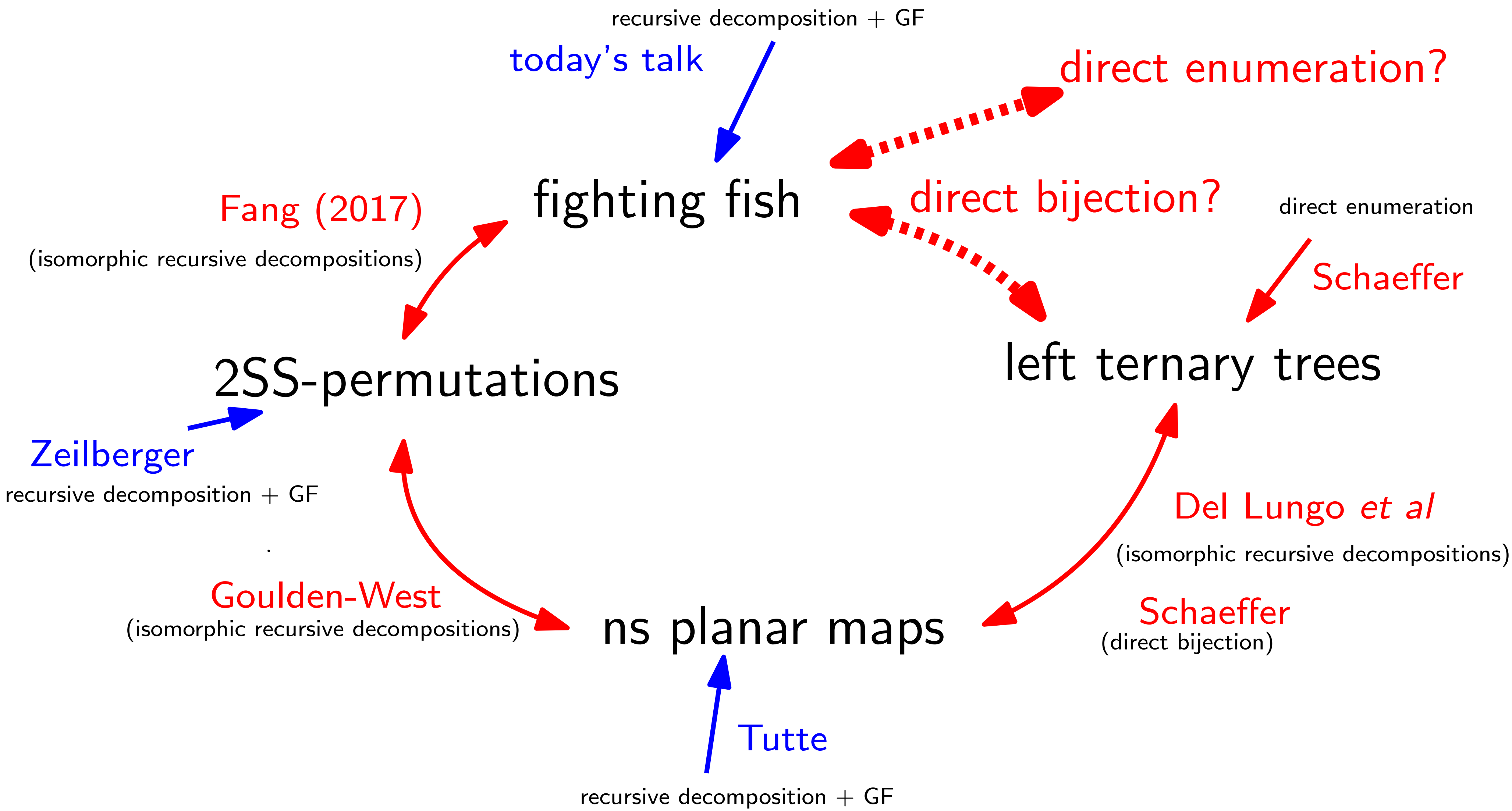
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Indeed, let us reconsider the equation for V, f, and A,

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Square root singularity expansion near dominant singularity

$$V = \frac{1}{3} - cte\sqrt{(1 - \frac{t}{t_c})} + O(1 - \frac{t}{t_c})$$

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The area generating function

We have the singular expansions:

$$f = \frac{1}{4} - \frac{3}{4}(1 - \frac{t}{t_c}) + \frac{\sqrt{3}}{2}(1 - \frac{t}{t_c})^{\frac{3}{2}} + O((1 - \frac{t}{t_c})^2)$$

$$A = 3 - cte \cdot (1 - \frac{t}{t_c})^{\frac{1}{4}} + O((1 - \frac{t}{t_c})^{\frac{1}{2}})$$

From transfert theorems: $g \sim (1 - \frac{t}{t_c})^{\alpha} \; \Rightarrow \; [t^n]g \sim \frac{n^{-1-\alpha}}{\Gamma(-\alpha)} t_c^{-n}$
we obtain:

$$[t^n]A \underset{n \to \infty}{\sim} cte \cdot n^{-\frac{5}{4}} t_c^{-n}$$

$$[t^n]f \underset{n \to \infty}{\sim} cte \cdot n^{-\frac{5}{2}} t_c^{-n}$$

Then the average area is

$$\frac{[t^n]A}{[t^n]f} \underset{n \to \infty}{\sim} cte \cdot n^{\frac{5}{4}}$$