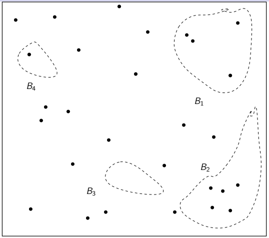
# Inhomogeneity cases in Poisson-Voronoi tessellations

**Pierre Calka**

*February 12, 2018*

Journées de combinatoire de Bordeaux

# Poisson point process



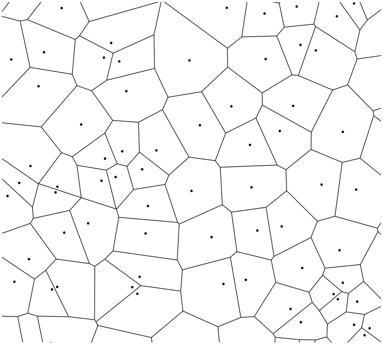

$\mathcal{P}_\lambda$ said Poisson point process in $\mathbb{R}^n$ with intensity $\lambda$ if

- $\#(\mathcal{P}_\lambda \cap B_1)$ Poisson r.v. of mean $\lambda \mathrm{vol}^{(\mathbb{R}^n)}(B_1)$
- $\#(\mathcal{P}_\lambda \cap B_1), \cdots, \#(\mathcal{P}_\lambda \cap B_\ell)$ independent $(B_1, \cdots, B_\ell \in \mathcal{B}(\mathbb{R}^n),\ B_i \cap B_j = \emptyset,\ i \neq j)$

*Two properties*

- Scaling invariance: $\mu \cdot \mathcal{P}_\lambda \overset{\mathcal{D}}{=} \mathcal{P}_{\frac{\lambda}{\sqrt[n]{\mu}}}$
- Mecke's formula: $\mathbb{E}\left( \sum_{x \in \mathcal{P}_\lambda} f(x, \mathcal{P}_\lambda) \right) = \lambda \int \mathbb{E}(f(x, \mathcal{P}_\lambda \cup \{x\}))\mathrm{d}x$

# Poisson-Voronoi tessellation



- $\mathcal{P}_\lambda$ homogeneous Poisson point process in $\mathbb{R}^n$ of intensity $\lambda$
- For every nucleus $x \in \mathcal{P}_\lambda$, associated cell

$$C(x, \mathcal{P}_\lambda) := \{y \in \mathbb{R}^2 : \|y - x\| \leq \|y - x'\| \; \forall x' \in \mathcal{P}_\lambda\}$$

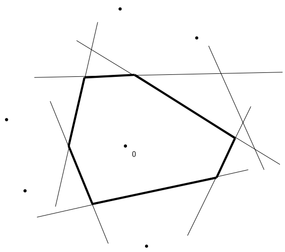
# Typical Poisson-Voronoi cell

- Typical cell $\mathcal{C}$ *chosen uniformly among all cells*

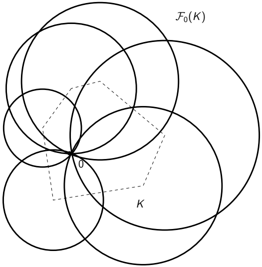
$$\mathbb{E}(f(\mathcal{C})) = \lim_{r\to\infty} \frac{1}{N_r} \sum_{\substack{x\in\mathcal{P}_\lambda \\ \{C(x,\mathcal{P}_\lambda)\subset B_r(o)}} f(C(x,\mathcal{P}_\lambda)) \text{ a.s.}$$

$$\mathbb{E}(f(\mathcal{C})) = \frac{1}{\lambda \mathrm{vol}^{(\mathbb{R}^n)}(B)} \mathbb{E}\Big( \sum_{x\in\mathcal{P}_\lambda\cap B} f(C(x,\mathcal{P}_\lambda) - x)\Big), \ B \in \mathcal{B}(\mathbb{R}^n)$$

- Theorem (Slivnyak) $\mathcal{C} \stackrel{D}{=} C(o, \mathcal{P}_\lambda \cup \{o\})$



# Being in the typical cell




- $K$ convex body containing $o$ in its interior
- Flower of $K$ $\quad \mathcal{F}_o(K) = \cup_{x\in K} B(x, \|x\|)$

*Two properties*

- Capacity probability: $\mathbb{P}(K \subset C(o, \mathcal{P}_\lambda \cup \{o\})) = e^{-\lambda \mathrm{vol}^{(\mathbb{R}^n)}(\mathcal{F}_o(K))}$
- Conditional distribution:
  $(\mathcal{P}_\lambda | K \subset C(o, \mathcal{P}_\lambda \cup \{o\})) \stackrel{D}{=} \mathcal{P}_\lambda \setminus \mathcal{F}_o(K)$

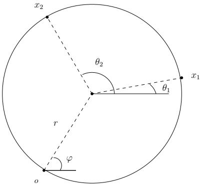
# Two mean calculations in $\mathbb{R}^2$

## Mean area

$$\mathbb{E}(\mathrm{vol}^{(\mathbb{R}^2)}(\mathcal{C})) = \int_{\mathbb{R}^2} \mathbb{P}(x \in C(o, \mathcal{P}_\lambda \cup \{o\})\mathrm{d}x = 2\pi \int_0^\infty e^{-\lambda\pi r^2} r\mathrm{d}r = \frac{1}{\lambda}$$

## Mean number of vertices

$\mathcal{N}(\cdot)$: number of vertices, $B(x,y,z) :=$ circumscribed disk of the triangle $xyz$




$$\begin{aligned}
\mathbb{E}(\mathcal{N}(\mathcal{C})) &= \mathbb{E} \sum_{\{x_1,x_2\}\subset\mathcal{P}_\lambda} \mathbf{1}_{\{B(o,x_1,x_2)\cap\mathcal{P}_\lambda=\emptyset\}} \\
&= \lambda^2 \int \mathbb{P}(B(o,x_1,x_2)\cap\mathcal{P}_\lambda=\emptyset)\mathrm{d}x_1\mathrm{d}x_2 \\
&= \lambda^2 \int e^{-\lambda \mathrm{vol}^{(\mathbb{R}^2)}(B(o,x_1,x_2))}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \lambda^2 \int e^{-\lambda\pi r^2} r^3 \mathrm{d}r \iint_{]0,2\pi[^2} \sin(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})|\sin(\frac{\theta_2-\theta_1}{2})|\mathrm{d}\theta_1\mathrm{d}\theta_2 \\
&= 6
\end{aligned}$$

# Plan

The planar Poisson-Voronoi cell around an isolated nucleus

Poisson-Voronoi cell in a Riemannian manifold

*Joint works with* **Aurélie Chapron**, **Yann Demichel** (Paris Nanterre) &
**Nathanaël Enriquez** (Paris-Sud)

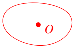
# Plan

# The planar Poisson-Voronoi cell around an isolated nucleus

- $K$ convex body in $\mathbb{R}^2$
- An origin $o$ chosen in $\mathrm{int}(K)$
- Point process $\mathcal{P}'_\lambda := (\mathcal{P}_\lambda | K \subset C(o, \mathcal{P}_\lambda \cup \{o\}))$
- *Aim* Asymptotics of the characteristics of the cell

$$K_\lambda = C(o, \mathcal{P}'_\lambda \cup \{o\})$$

# The planar Poisson-Voronoi cell around an isolated nucleus

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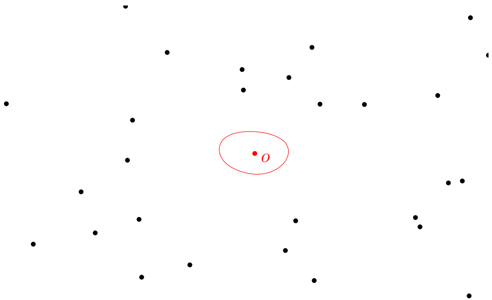


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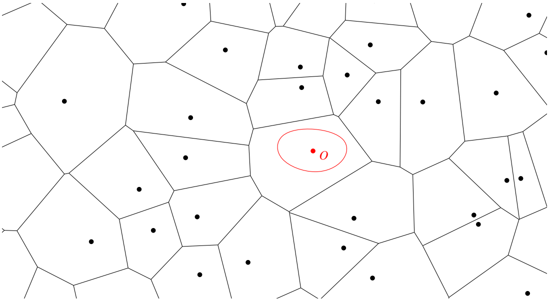


- $K$ convex body in $\mathbb{R}^2$
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- *Aim* Asymptotics of the characteristics of the cell

$$K_\lambda = C(o, \mathcal{P}'_\lambda \cup \{o\})$$

# Context: large Poisson-Voronoi cells

- Large cells in a Poisson-Voronoi tessellation: *close* to the circular shape

  **D. Hug, M. Reitzner & R. Schneider** (2004)

- When $K$ is the unit-disk,

  $$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-1} 3^{-\frac{1}{3}} \pi \Gamma\left(\frac{2}{3}\right), \ \mathbb{E}(\mathcal{N}(K_\lambda)) \underset{\lambda\to\infty}{\sim} \lambda^{\frac{1}{3}} 2^{2} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right)$$

  **PC & T. Schreiber** (2005)

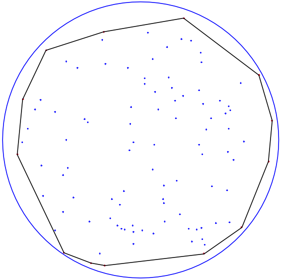
- Estimate of the Hausdorff distance between $K_\lambda$ and $K$ for a Poisson line tessellation

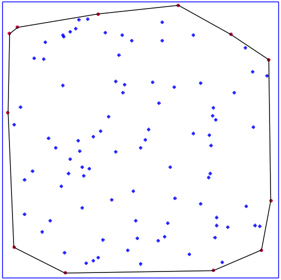
  **D. Hug & R. Schneider** (2014), **R. Schneider** (1988)

# Context: approximation of a convex body from the inside

$$K^\lambda = \mathrm{Conv}(\mathcal{P}_\lambda \cap K)$$

Efron's relation (1965) $\mathbb{E}(\mathcal{N}(K^\lambda)) = \lambda(\mathcal{A}(K) - \mathbb{E}(\mathcal{A}(K^\lambda)))$





$K$ with a smooth boundary

$$\mathcal{A}(K) - \mathbb{E}(\mathcal{A}(K^\lambda)) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{\frac{4}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{1}{3}} \mathrm{d}s$$

$K$ polygon

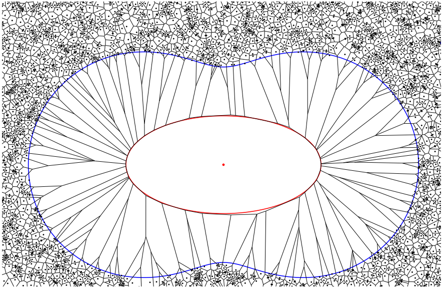
$$\mathcal{A}(K) - \mathbb{E}(\mathcal{A}(K^\lambda)) \underset{\lambda\to\infty}{\sim} (\lambda^{-1}\log\lambda) \cdot 2 \cdot 3^{-1} n_K$$

**A. Rényi & R. Sulanke** (1963, 1964)

# Main results: smooth case

$\mathcal{A}(\cdot)$: area, $\mathcal{U}(\cdot)$: perimeter, $\mathcal{N}(\cdot)$: number of vertices

$r_s$: radius of curvature, $n_s$: outer unit normal vector at $s \in \partial K$



$$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-2} 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{\frac{1}{3}} \langle s, n_s\rangle^{-\frac{2}{3}} \, \mathrm{d}s$$

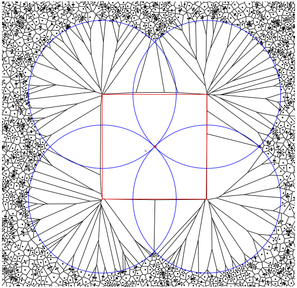
$$\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s\rangle^{-\frac{2}{3}} \mathrm{d}s$$

$$\mathbb{E}(\mathcal{N}(K_\lambda)) \underset{\lambda\to\infty}{\sim} \lambda^{\frac{1}{3}} 2^2 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s\rangle^{\frac{1}{3}} \, \mathrm{d}s$$

# Main results: polygonal case

$\mathcal{A}(\cdot)$: area, $\mathcal{U}(\cdot)$: perimeter, $\mathcal{N}(\cdot)$: number of vertices

$n_K$: number of vertices of $K$, $\{a_i\}$: vertices of $K$, $o_i$: projection of $o$ onto $(a_i, a_{i+1})$

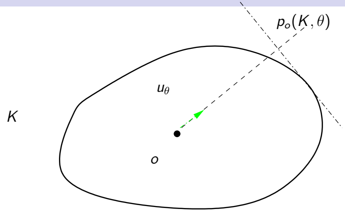


$$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{1}{2}} 2^{-\frac{9}{2}} \pi^{\frac{3}{2}} \sum_{i=1}^{n_K} \|o_i\|^{-\frac{1}{2}} \|a_{i+1} - a_i\|^{\frac{3}{2}}$$

$$\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K) \underset{\lambda\to\infty}{\sim} (\lambda^{-1}\log\lambda) \cdot 2^{-1} 3^{-1} \textstyle\sum_{i=1}^{n_K} \|o_i\|^{-1}$$

$$\mathbb{E}(\mathcal{N}(K_\lambda)) \underset{\lambda\to\infty}{\sim} (\log\lambda) \cdot 2 \cdot 3^{-1} n_K.$$

# Sketch of proof: support function




## Support function

$$p_o(K,\theta) = p_o(K,u_\theta) = \sup_{x\in K}\langle x, u_\theta\rangle$$

with $u_\theta = (\cos(\theta), \sin(\theta))$

### *Two properties*

- Link with the flower: $\sup\{r > 0 : ru_\theta \in \mathcal{F}_o(K)\} = 2p_o(K,\theta)$
- Cauchy-Crofton formula: $\mathcal{U}(K) = \int_0^{2\pi} p_o(K,\theta)\mathrm{d}\theta$

# Sketch of proof: rewriting of the expectations

- Mean defect area

$$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) = \int_{\mathbb{R}^2\setminus K} \mathbb{P}(x \in K_\lambda)\mathrm{d}x$$
$$= \int_{\mathbb{R}^2\setminus K} e^{-\lambda(\mathcal{A}(\mathcal{F}_o(K\cup\{x\}))-\mathcal{A}(\mathcal{F}_o(K)))}\mathrm{d}x$$

where $\mathcal{A}(\mathcal{F}_o(K)) = 2\int_0^{2\pi} p_o(K,\theta)^2\mathrm{d}\theta$

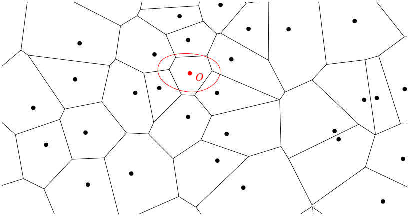
- Mean defect perimeter

$$\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K) = \int_0^{2\pi} \mathbb{E}(p_o(K_\lambda,\theta) - p_o(K,\theta))\mathrm{d}\theta$$

- Mean number of vertices Efron-type relation

$$\mathbb{E}(\mathcal{N}(K_\lambda)) = \lambda(\mathbb{E}(\mathcal{A}(\mathcal{F}_o(K_\lambda)) - \mathcal{A}(\mathcal{F}_o(K)))$$
$$\underset{\lambda\to\infty}{\sim} 4\lambda\int_0^{2\pi} p_o(K,\theta)\mathbb{E}(p_o(K_\lambda,\theta) - p_o(K,\theta))\mathrm{d}\theta$$

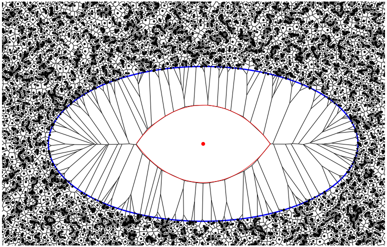
# Second model

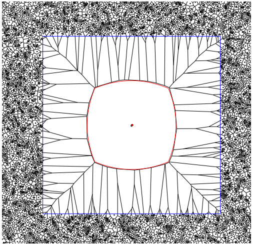



- $\mathcal{D}$ closed domain, $o \in \mathrm{int}(\mathcal{D})$
- Point process $\mathcal{P}'_\lambda := (\mathcal{P}_\lambda | \mathcal{D} \cap \mathcal{P}_\lambda = \emptyset\}$
- *Aim* Asymptotics of the mean characteristics of the cell

$$\mathcal{C}_\lambda(\mathcal{D}) = C(o, \mathcal{P}'_\lambda \cup \{o\})$$

# Result for the second model





- $K$: convex body such that $\mathcal{F}_o(K)$ is the largest flower included in $\mathcal{D}$
- $\mathcal{D}^*$: maximal starlike set included in $\mathcal{D}$ with a polar equation $d(\cdot)$ piecewise $C^3$

$$\mathcal{C}_\lambda(\mathcal{D}) \xrightarrow{\mathbb{P}} K \text{ pour la métrique de Hausdorff}$$

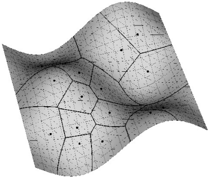
$$\mathbb{E}(\mathcal{A}(\mathcal{C}_\lambda(\mathcal{D}))) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-\frac{8}{3}} 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) \int (d(\theta) + d''(\theta))^{\frac{4}{3}} d(\theta)^{-\frac{2}{3}} \mathrm{d}\theta$$

$$\mathbb{E}(\mathcal{U}(\mathcal{C}_\lambda(\mathcal{D}))) - \mathcal{U}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int (d(\theta) + d''(\theta))^{\frac{1}{3}} d(\theta)^{-\frac{2}{3}} \mathrm{d}\theta$$

$$\mathbb{E}(\mathcal{N}(\mathcal{C}_\lambda(\mathcal{D}))) \underset{\lambda\to\infty}{\sim} \lambda^{\frac{1}{3}} 2^{-\frac{8}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int (d(\theta) + d''(\theta))^{\frac{1}{3}} d(\theta)^{\frac{1}{3}} \mathrm{d}\theta$$

# Plan

# Model



© **R. Kunze** (1985)

- $M$ Riemannian manifold endowed with the distance $d$ and the volume measure $\mathrm{vol}^{(M)}$
- $\mathcal{P}_\lambda$ Poisson point process in $M$ of intensity measure $\lambda \mathrm{vol}^{(M)}$
- Voronoi cell associated with $x \in \mathcal{P}_\lambda$

$$C^{(M)}(x, \mathcal{P}_\lambda) = \{y \in M : d(x,y) \leq d(x',y) \, \forall x' \in \mathcal{P}_\lambda\}$$

*Aim* Study the mean geometrical characteristics of

$$\mathcal{C}^{(M)}_{x_o,\lambda} := C^{(M)}(x_0, \mathcal{P}_\lambda \cup \{x_0\}), \quad x_0 \in M \text{ fixed}$$

# Context

- Influence of the local geometry of $M$ around $x_0$ on the geometry of $C^{(M)}(x_0, \mathcal{P}_\lambda \cup \{x_0\})$

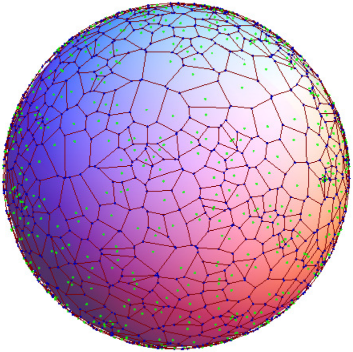
  $\rightsquigarrow$ *Need to do estimations at high intensity*

- Conversely, information provided by the tessellation on the geometry of $M$, both local (curvatures at $x_0$) and global (Gauss-Bonnet)

  $\rightsquigarrow$ *Need to show limit theorems in order to do a statistical estimation*

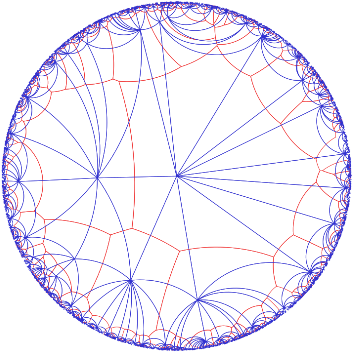
# Previous works

$\mathcal{S}_k^2$
2-sphere of radius $1/k$



© **J. Diaz-Polo & I. Garay** (2014)

$\mathcal{H}_k^2$
hyperbolic plane of curvature $-k^2$



© **I. Benjamini, E. Paquette & J. Pfeffer** (2014)

# Previous works

$\mathcal{S}_k^2$
2-sphere of radius $1/k$

$\mathcal{H}_k^2$
hyperbolic plane of curvature $-k^2$

**Mean area**

$$\mathbb{E}(\mathrm{vol}^{(\mathcal{S}_k^2)}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{S}_k^2)})) = \frac{1}{\lambda}\left(1 - e^{-\frac{4\pi\lambda}{k^n}}\right)$$

**Mean number of vertices**

$$\mathbb{E}\left(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{S}_k^2)})\right)$$
$$= 6 - \frac{3k^2}{\pi\lambda} + e^{-\frac{4\pi\lambda}{k^2}}\left(6 + \frac{3k^2}{\pi\lambda}\right)$$

**R. E. Miles** (1971)

**Mean area**

$$\mathbb{E}(\mathrm{vol}^{(\mathcal{H}_k^n)}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{H}_k^2)})) = \frac{1}{\lambda}$$

**Mean number of vertices**

$$\mathbb{E}\left(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{H}_k^2)})\right) = 6 + \frac{3k^2}{\pi\lambda}.$$

**Y. Isokawa** (2000)

# Main result

$\mathrm{Sc}_{x_0}^{(M)} :=$ scalar curvature of $M$ at $x_0$

When $\lambda \to \infty$,

$$\mathbb{E}(\mathrm{vol}^{(M)}(\mathcal{C}_{x_0,\lambda}^{(M)})) = \frac{1}{\lambda} + o\left(\frac{1}{\lambda^{1+\frac{2}{n}}}\right)$$

$$\mathbb{E}(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(M)})) = e_n - d_n \,\mathrm{Sc}_{x_0}^{(M)} \frac{1}{\lambda^{\frac{2}{n}}} + o\left(\frac{1}{\lambda^{\frac{2}{n}}}\right)$$

where $e_n = \mathbb{E}(\mathcal{N}(C^{(\mathbb{R}^n)}(o, \mathcal{P}_\lambda \cup \{o\}))$ and $d_n$ are explicit.

# Sketch of proof: preliminary calculation

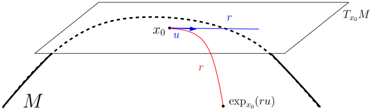
$$\begin{aligned}
&\mathbb{E}(\mathcal{N}(\mathcal{C}^{(M)}_{x_0,\lambda}))\\
&= \mathbb{E} \sum_{\{x_1,\dots,x_n\}\subset\mathcal{P}_\lambda} \sum_{\mathcal{B} \text{ circumball of } x_0,\dots,x_n} \mathbf{1}_{\{\mathcal{B}\cap\mathcal{P}_\lambda=\emptyset\}}\\
&\approx \lambda^n \int_{\mathcal{B}^{(M)}(x_0,\varepsilon)^n} \mathbb{P}(\text{CircumBall}(x_0,\dots,x_n)\cap\mathcal{P}_\lambda=\emptyset)\mathrm{d}\mathsf{vol}^{(M)}(x_1)\dots\mathrm{d}\mathsf{vol}^{(M)}(x_n)\\
&= \lambda^n \int_{\mathcal{B}^{(M)}(x_0,\varepsilon)^n} e^{-\lambda \mathsf{vol}^{(M)}(CircumBall(x_0,\dots,x_n))}\mathrm{d}\mathsf{vol}^{(M)}(x_1)\dots\mathrm{d}\mathsf{vol}^{(M)}(x_n)
\end{aligned}$$

*Aim* Estimates when $x_1,\dots,x_n \to x_0$ of

$\mathsf{vol}^{(M)}(CircumBall(x_0,\dots,x_n))$ and $\mathrm{d}\mathsf{vol}^{(M)}(x_1)\dots\mathrm{d}\mathsf{vol}^{(M)}(x_n)$

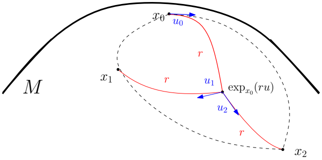
before applying Laplace's method

# Sketch of proof: change of variables




Exponential map

For each unit-vector $u \in T_{x_0}M$, $\gamma : r \longmapsto \exp_{x_0}(ru)$ is the unique geodesic emanating from $x_0$ with speed 1 and $\gamma'(0) = u$.




Change of variables $\quad x_i = \exp_{|\exp_{x_0}(ru_0)}(ru_i),\ i = 1, \dots, n$

# Sketch of proof: asymptotics for the integrand

- Expansion of the Jacobian

$$\mathrm{dvol}^{(M)}(x_1)\ldots\mathrm{dvol}^{(M)}(x_n)=\mathcal{J}\,\mathrm{d}r\,\mathrm{dvol}^{(\mathbb{S}^{n-1})}(u_0)\ldots\mathrm{dvol}^{(\mathbb{S}^{n-1})}(u_n),$$

$$\mathcal{J}=n!\Delta(u_0,\ldots,u_n)\left(r^{n^2-1}-\frac{\sum_{i=0}^{n}\mathrm{Ric}_{x_0}^{(M)}(u_i)}{6}r^{n^2+1}+o(r^{n^2+1})\right)$$

$\mathrm{Ric}_{x_0}^{(M)}(u_i)$:= Ricci curvature at $x_0$ in direction $u_i$, $\Delta(u_0,\ldots,u_n)$:= $\mathrm{vol}^{(\mathbb{R}^n)}(\mathrm{Conv}(u_0,\ldots,u_n))$

- Volume of small balls

Theorem (Bertrand-Diguet-Puiseux)

$$\mathrm{vol}^{(M)}(\mathcal{B}^{(M)}(x_0,r))=\mathrm{vol}^{(\mathbb{R}^n)}(\mathcal{B}^{(\mathbb{R}^n)}(o,r))\left(1-\frac{\mathrm{Sc}_{x_0}^{(M)}}{6(n+2)}r^2+o(r^2)\right)$$

# Limit theorems and estimation: expectation and variance

*Aim* Replace $\mathbb{E}(\mathcal{N}(\mathcal{C}^{(M)}_{x_0,\lambda}))$ with an empirical mean in the vicinity of $x_0$

$$N^{(M)}_\lambda := \sum_{y \in B(x_0,\lambda^{-\beta})\cap \mathcal{P}_\lambda} \mathcal{N}(C^{(M)}(y,\mathcal{P}_\lambda)), \quad 0<\beta<1/n$$

$$\mathbb{E}(N^{(M)}_\lambda) \underset{\lambda\to\infty}{=} \kappa_n \lambda^{1-n\beta}(e_n - d_n \, \mathrm{Sc}^{(M)}_{x_0}\, \lambda^{-\frac{2}{n}}) + o(\lambda^{1-n\beta-\frac{2}{n}})$$

$$\mathrm{Var}(N^{(M)}_\lambda) \underset{\lambda\to\infty}{\sim} \kappa_n \lambda^{1-n\beta}\sigma_\infty$$

where $\kappa_n :=$ volume of the $n$-dimensional unit-ball
and $\sigma_\infty := \mathbb{E}(\mathcal{N}(\mathcal{C}^{(\mathbb{R}^n)}_{o,1})^2) + \int \mathrm{Cov}(\mathcal{N}(\mathcal{C}^{(\mathbb{R}^n)}(x,\mathcal{P}_1\cup\{x\})), \mathcal{N}(\mathcal{C}^{(\mathbb{R}^n)}_{o,1}))\mathrm{d}x.$

$\rightsquigarrow$ $(N^{(M)}_\lambda - \mathbb{E}(N^{(M)}_\lambda))$ of order $\sqrt{\mathrm{Var}(N^{(M)}_\lambda)}$, i.e. $\lambda^{\frac{1}{2}(1-n\beta)}$
must be negligible in front of the second term of $\mathbb{E}(N^{(M)}_\lambda))$, i.e. $\lambda^{1-n\beta-\frac{2}{n}}$.

# Limit theorems and estimation: central limit theorem

$0 < \beta < \frac{1}{n} - \frac{4}{n^2}$ and $n \geq 5$

$N_\lambda^{(M)} := \sum_{y \in B(x_0, \lambda^{-\beta}) \cap \mathcal{P}_\lambda} \mathcal{N}(C^{(M)}(y, \mathcal{P}_\lambda))$

- $\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left( \frac{N_\lambda^{(M)} - \mathbb{E}(N_\lambda^{(M)})}{\sqrt{\mathrm{Var}(N_\lambda^{(M)})}} \leq t \right) - \mathbb{P}(\mathcal{N}(0,1) \leq t) \right| \leq c \lambda^{\frac{1}{2}(1-n\beta)}$.

- $\widehat{\mathrm{Sc}}_\lambda(x_0) := \frac{\lambda^{\frac{2}{n}}}{d_n}(e_n - \frac{1}{\kappa_n \lambda^{1-n\beta}} N_\lambda^{(M)})$

is an asymptotically unbiased, consistent and normal estimator of $\mathrm{Sc}_{x_0}^{(M)}$.

# Sketch of proof

- *Law of large numbers* We require the uniformity of the expansion of $\mathbb{E}(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(M)}))$ with respect to $x_0$.
- *Limiting variance* Each Voronoi cell is influenced by the points at distance of order $\lambda^{-\frac{1}{n}}$ with high probability.
- *Limiting variance 2* We apply the inverse of the exponential map and a rescaling, then compare the tessellation with an Euclidean Voronoi tessellation in $\mathbb{R}^n$.
- *Central limit theorem* We use Stein's method and estimates of the Malliavin derivatives for the central limit theorem.

  **G. Last, G. Peccati & M. Schulte** (2016)

Thank you for your attention!