Integrable statistical mechanics on isoradial graphs

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joint work with Béatrice de Tilière, K. Raschel

Outline

Integrability

Isoradial graphs

Statistical mechanics at criticality

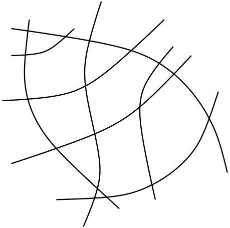
Out of criticality

Integrability

Integrability

Additional algebraic structure in the model allowing (much) more explicit computations than in generic situations

Example: 6-vertex model on 4-regular planar graph



configurations: orientations such that two in/out going edges at each vertex

Integrability in statistical mechanics

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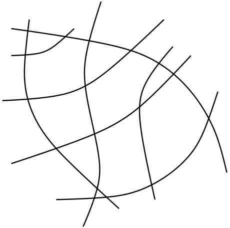
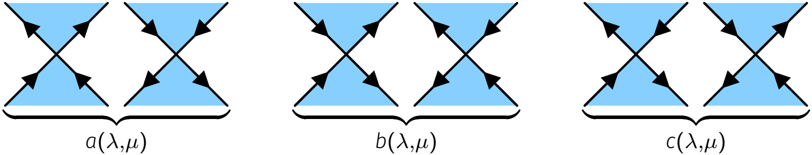
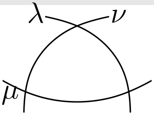


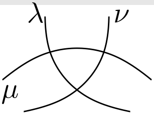
configurations: orientations such that two in/out going edges at each vertex

weights:



Integrability of the 6-vertex model

integrable if: $\sum$ 

$= \sum$ 

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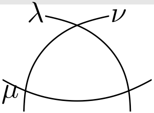
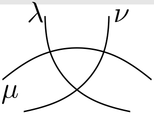
integrable if: $\sum$ 

$= \sum$ 

Encode the weights in a matrix

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

Yang–Baxter equation:

$$R_{12}(\lambda, \mu)R_{13}(\lambda, \nu)R_{23}(\mu, \nu) = R_{23}(\mu, \nu)R_{13}(\lambda, \nu)R_{12}(\lambda, \mu)$$

Corresponds to a *zero curvature condition*

Definition

$f : \mathbb{R}^m \to \mathbb{R}^N$ is a conjugate net if

$$\forall\, u,\ \forall\, i \neq j, \quad \partial_i \partial_j f \in \mathrm{span}(\partial_i f, \partial_j f)$$

Integrability in differential geometry

Corresponds to a *zero curvature condition*

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$$\forall\, u,\ \forall\, i \neq j, \exists\, c_{ij}, c_{ji}, \quad \partial_i \partial_j f = c_{ji} \partial_i f + c_{ji} \partial_j f$$

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f and f^+ are related by a F-transformation if

$$\forall\, u,\ \forall\, i, \quad \partial_i f, \partial_i f^+, f^+ - f \text{ coplanar}$$

Geometric integrability and consistency

Permutability of F-transformations:

$$\begin{tikzcd}
f^2 & \\
f \arrow[u] \arrow[r] & f^1
\end{tikzcd}$$

Geometric integrability and consistency

Permutability of F-transformations:

$$
\begin{tikzcd}
f^2 \arrow[r] & f^{12} \\
f \arrow[u] \arrow[r] & f^1 \arrow[u]
\end{tikzcd}
$$

Geometric integrability and consistency

Permutability of F-transformations:

$$
\begin{tikzcd}
f^2 \arrow[rr] & & f^{12} \\
& f^3 & \\
f \arrow[uu] \arrow[ur] \arrow[rr] & & f^1 \arrow[uu]
\end{tikzcd}
$$

Geometric integrability and consistency

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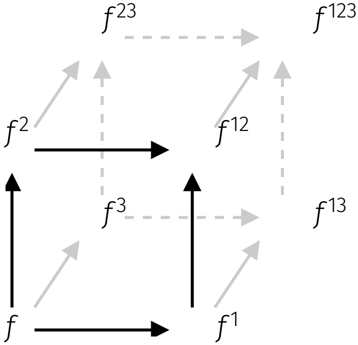

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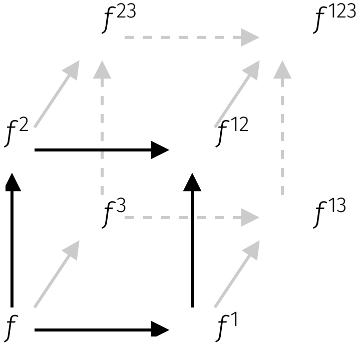
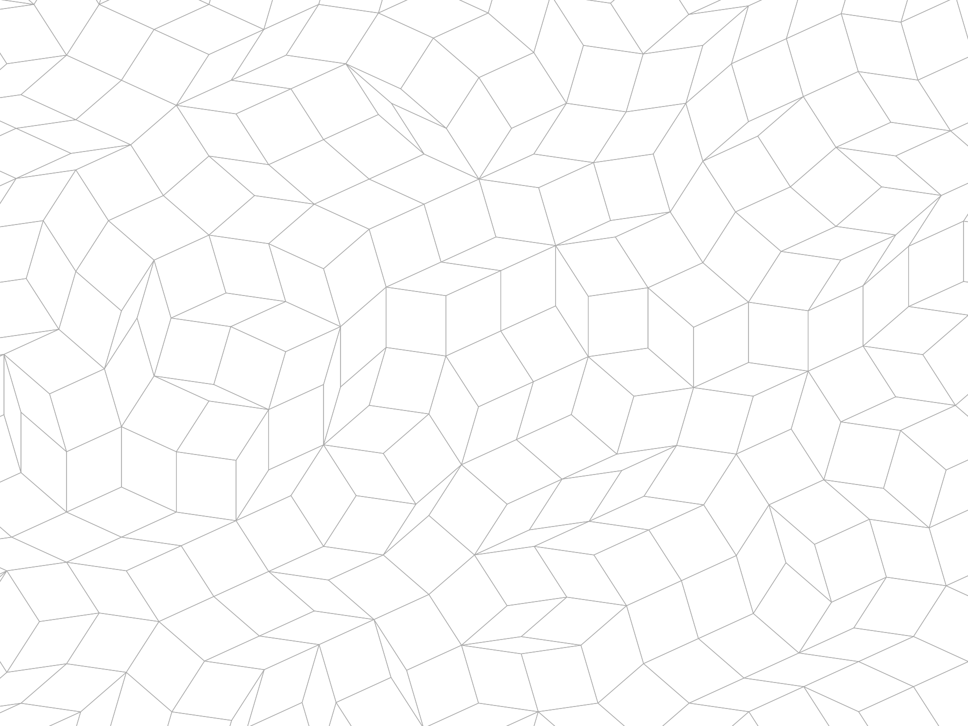
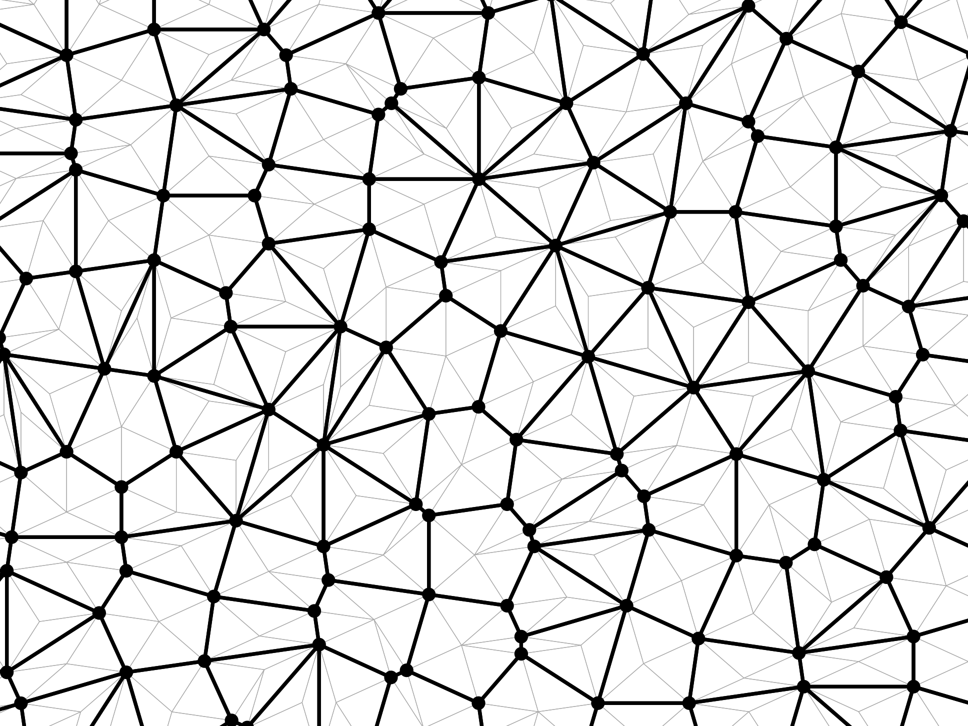


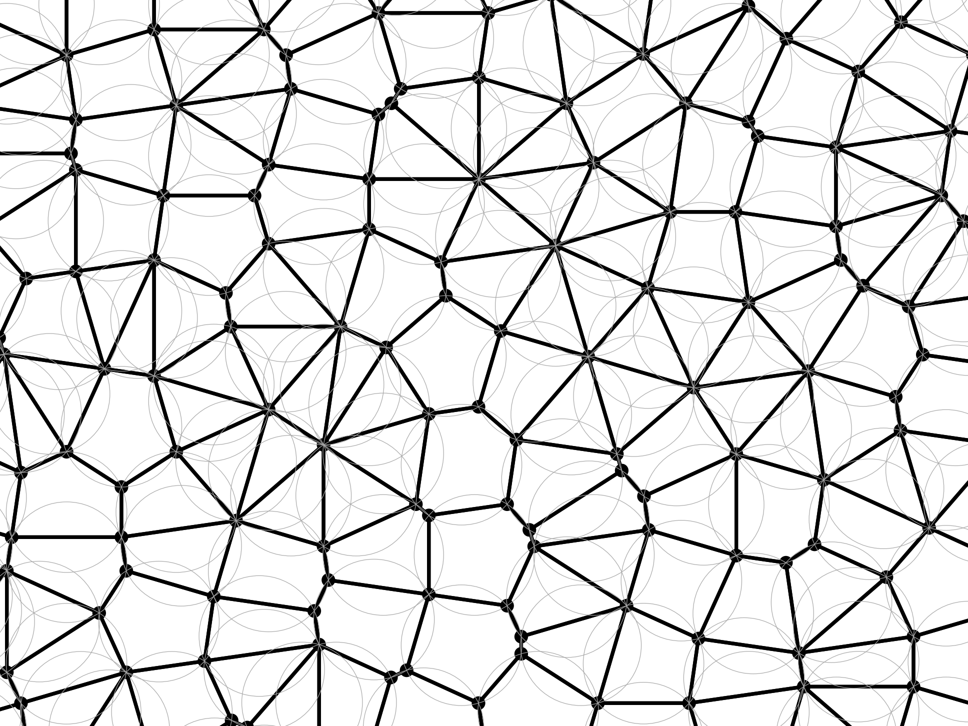
Discrete differential geometry

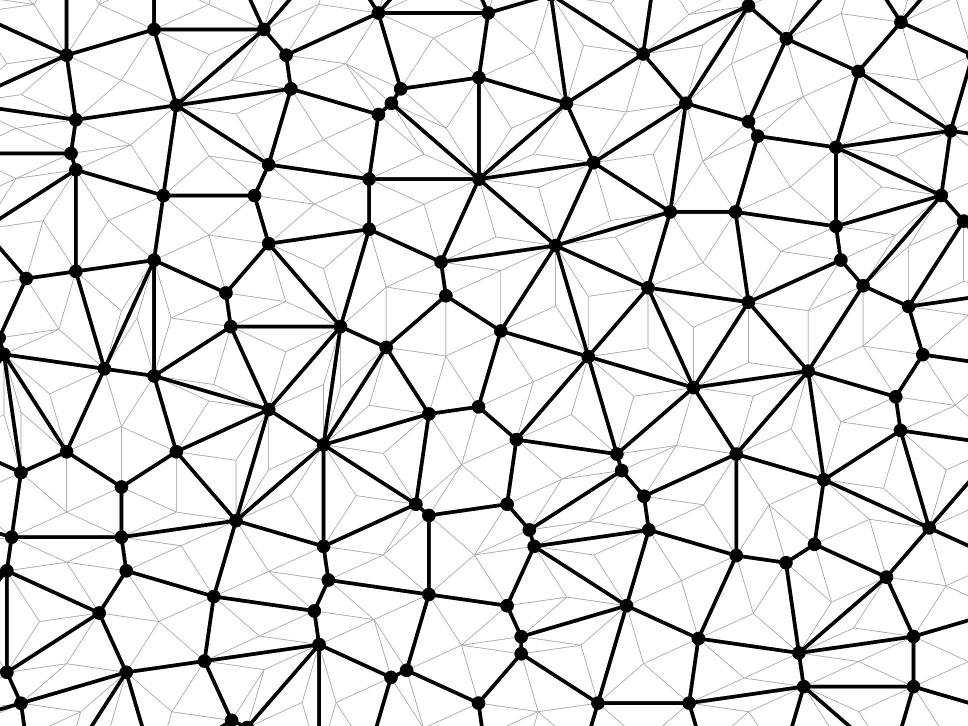
Discretize not only the equation but the whole theory (Bobenko–Suris)

Isoradial graphs



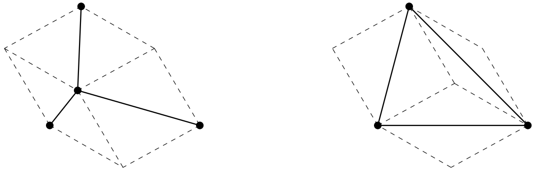






Isoradial graphs

- quad graph : projection of a surface in $\mathbb{Z}^d$ (discrete net)
- natural flip operation



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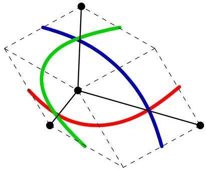

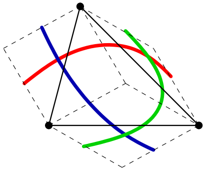

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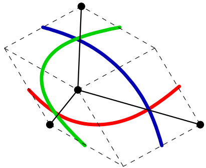

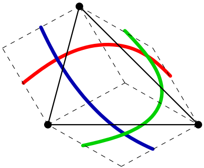

- Two isoradial graphs coinciding outside a ball are obtained one from the other with flips (Kenyon)

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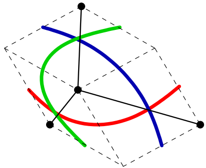
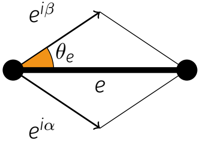


- Two isoradial graphs coinciding outside a ball are obtained one from the other with flips (Kenyon)
- Each edge e has a natural parameter $\theta_e = \frac{\beta-\alpha}{2}$



Integrability, 3D consistency and star-triangle transformation

$$\Delta f(x) = \sum_{e:y\sim x} c(\theta_e)(f(x) - f(y))$$

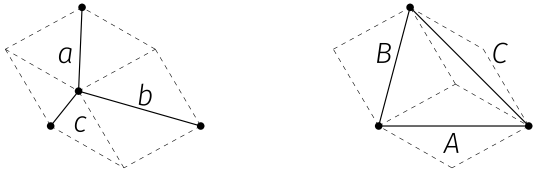
3D consistency of harmonic functions

G, G': isoradial graphs differing by a star-triangle transformation. (G' has one more vertex)

- The restriction to G of a harmonic function on G' is harmonic on G
- Every harmonic function on G can be extended uniquely to G' so that the extension is harmonic on G'.

Q: find function c s.t the equation $\Delta f = 0$ is 3D-consistent?

Integrability, 3D consistency and star-triangle transformation



Electrical networks (Kennelly)

if require equivalence of electric, and impose $a = 1/A, \dots$

$$a + b + c = a \cdot b \cdot c$$

parameterized by $\theta + \varphi + \psi = \pi$:

$$\tan(\theta) + \tan(\varphi) + \tan(\psi) = \tan(\theta)\tan(\varphi)\tan(\psi)$$

3-dimensional consistency

$$\Delta f(x) = \sum_{e:y\sim x} \tan(\theta_e)(f(x) - f(y))$$

Lemma (3D consistency of $\Delta f = 0$)

Let G and G' be two isoradial graphs differing by a star-triangle transformation. (G' has one more vertex)

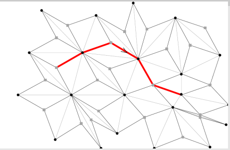
- *The restriction to G of a harmonic function on G' is harmonic on G*
- *Every harmonic function on G can be extended uniquely to G' so that the extension is harmonic on G'.*

$\rightsquigarrow$ transport of harmonic functions

discrete exponential functions

Definition (Discrete exponential function, Mercat)

$$\mathrm{Exp}_{x,y}(\lambda) = \prod_{j=1}^{n} \frac{\lambda - e^{i\alpha_j}}{\lambda + e^{i\alpha_j}}, \quad \lambda \in \mathbb{C} \cup \{\infty\}$$



Lemma

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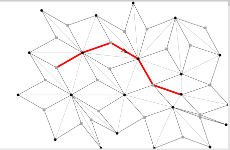

Lemma

Discrete exponential functions are harmonic

If $\lambda = e^{iu}$, $\frac{\lambda - e^{i\alpha}}{\lambda + e^{i\alpha}} = i \tan \frac{u-\alpha}{2}$ and $f(x) = \mathrm{Exp}_{x,y}(\lambda)$.

$$\begin{aligned}
&\tan(\theta)(f(x) - f(y)) \\
&\qquad = \tan\left(\frac{\beta - \alpha}{2}\right)\left(1 + \tan\left(\frac{u-\alpha}{2}\right)\tan\left(\frac{u-\beta}{2}\right)\right) \\
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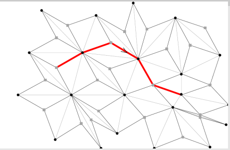

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telescopic sum

Green function

Theorem (Kenyon)

Green function between two vertices x and y:

$$G(x,y) = -\frac{1}{8i\pi^2} \oint \mathrm{Exp}_{x,y}(\lambda) \log(\lambda) \frac{d\lambda}{\lambda}$$

- locality : $G(x,y)$ depends only on the geometry of a path between x and y
- integral formula: suitable for asymptotics

Statistical mechanics at criticality

Models directly associated to the Laplacian

Random walk associated to conductances $\tan(\theta)$

- martingale
- scaling limit: (time changed) standard Brownian motion

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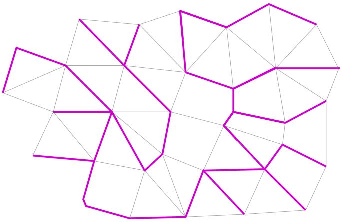
Spanning trees



weight$(T) = \prod_{e \in T} c(e)$

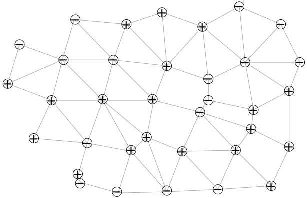
Gibbs measure (Burton-Pemantle): determinantal process on edges with kernel given by the **transfer impedance matrix** :

$$H(e,f) = c(e)\big(\mathcal{G}(e^+,f^+) - \mathcal{G}(e^+,f^-) - \mathcal{G}(e^-,f^+) - \mathcal{G}(e^-,f^-)\big)$$

Critical integrable Ising model

- coupling constants $J_e > 0$
- spin configurations σ: ± 1 at vertices
- energy $\mathcal{H}(\sigma) = -\sum_{e=xy} J_e \sigma_x \sigma_y$
- probability of a configuration

$$\mathbb{P}(\sigma) = \frac{1}{Z_{\text{Ising}}(G)} e^{-\mathcal{H}(\sigma)}$$



- integrability + autoduality [Baxter]

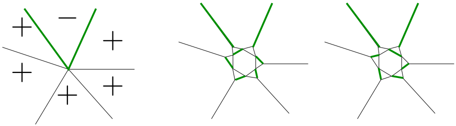
$$J(\theta) = \frac{1}{2} \log\left(\frac{1+\cos(\theta)}{\sin(\theta)}\right) \quad \text{or} \quad \sinh(2J(\theta)) = \tan(\theta)$$

Critical integrable Ising model

- really critical [Li, Cimasoni–Duminil-Copin]
- discrete harmonic fermionic observable
- conformally invariant scaling limit [Chelkak–Smirnov,...]
- construction of probability measure in infinite volume on isoradial graphs (dimers, Fisher correspondance) [B.–de Tilière]
- locality of spin correlations

Ising model (on faces) and dimers: Fisher's bijection

- contours separating spin connec. comp. (low temperature)



- K signed adjacency matrix on the decorated graph
- K^{-1} : contour integral of exponential functions

Theorem (Kasteleyn)

If $e_1 = (v_1, v_2)$,...,$e_k = (v_{2k-1}, v_{2k})$, the probability to see e_1,...,e_k in a random dimer configuration is

$$\mathbb{P}(e_1, \ldots, e_k) = \left(\prod_{j=1}^{k} K(v_{2j-1}, v_{2j}) \right) \mathrm{Pfaff}_{i,j}\, {}^tK^{-1}(v_i, v_j)$$

Out of criticality

What can we say about models out of criticality?

- parameter to study the phase transition?
- keep integrability (and locality?)
- scaling limit in the near critical regime

Non critical integrable Ising model

invariance of the Ising model under the star-triangle transformation $\Rightarrow$ one parameter family of coupling constants

$$\sinh 2J(\theta|k) = \mathrm{sc}\left(\frac{2K(k)\theta}{\pi}|k\right)$$

- k elliptic modulus: plays the role of temperature
- $k = 0$: critical (elliptic functions $\to$ trigonometric)
- $K(k)$ elliptic integral of 1st kind

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \cos^2(s)}} ds$$

- sc elliptic version of $\tan$

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What is the replacement for the exponential functions?

Integrable massive Laplacian

(with B. de Tilière and K. Raschel)

$$\Delta^m f(v) = \sum_{e:w\sim v} c(\theta_e)(f(v) - f(w)) + m_v f(v)$$

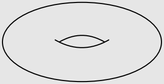
conductances : $c(\theta) = \mathrm{sc}(\theta|k)$

mass : $m_v = \sum_{e:w\sim v} A(\theta_e|k) - \mathrm{sc}(\theta_e|k)$

A, sc: elliptic functions. k: elliptic modulus, $k' = \sqrt{1-k^2}$

Definition (massive exponential functions)

$$\mathrm{Exp}_{x,y}(u|k) = \prod_j i\sqrt{k'}\,\mathrm{sc}\Big(\frac{u-\alpha_j}{2}|k\Big),\ u \in T_k$$



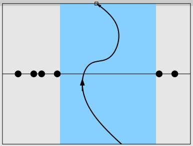
Massive exponential functions and Green function

Lemma (B–de Tilière–Raschel)

- *The equation* $\Delta^m f = 0$ *is 3D consistent*
- $x \mapsto \mathrm{Exp}_{x,y}(u|k)$ *is massive harmonic*

Theorem (B–de Tilière–Raschel)

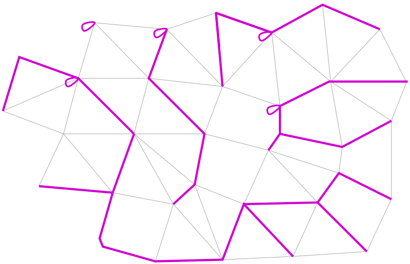
$$G^m_{x,y} = \frac{k'}{4i\pi} \oint_{C_{x,y}} \mathrm{Exp}_{x,y}(u|k)du$$



- Local expression
- Explicit exponential decay (saddle point analysis)

Massive Laplacian and rooted spanning forests

On a finite graph, the determinant of a massive Laplacian "counts" the rooted spanning forests



weight of a forest F :

$$w(F) = \prod_{T \subset F} m_{v_T} \prod_{e \in T} \rho(e)$$

$$\det \Delta^m = \sum_F w(F)$$

correlations : determinantal process
with kernel : the transfer impedance
(difference of Green functions)

Rooted spanning forests on an isoradial graph

- construction of a Gibbs measure for forests on infinite isoradial graph (locality of the transfer impedance)
- free energy of the model as a function of the geometry of the embedding

$$F(k) = -|V_1| \int_0^K 4H'(2\theta) \log \mathrm{sc}(\theta) d\theta - \sum_{e \in E_1} \int_0^{\theta_e} \frac{2H(2\theta)\, \mathrm{sc}'(\theta)}{\mathrm{sc}(\theta)} d\theta$$

- phase transition to spanning trees when $k \to 0$

$$F(k) = F_{\mathrm{trees}} - |V_1| k^2 \log k^{-1} + O(k^2)$$

Many probabilistic properties connected to geometric properties of underlying algebraic curves

Spectral curves

We suppose now that the isoradial graph is periodic

- Δ^m periodic $\rightsquigarrow$ Fourier transform $\widehat{\Delta^m}(z,w)$
- spectral curve : zeros de $\det \widehat{\Delta^m}(z,w)$ in $(\mathbb{C}^*)^2$

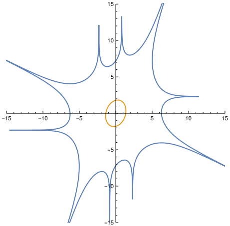
Theorem (B–de Tilière–Raschel)

spectral curve of a massive Laplacian on isoradial graph

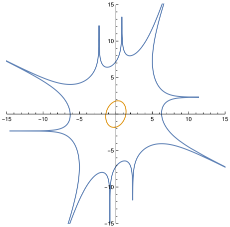
$$\Updownarrow$$

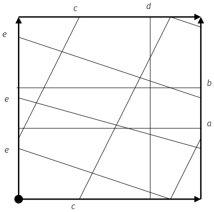
Harnack curve of genus 1 with symmetry

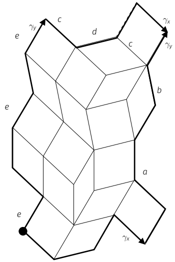
$$(z,w) \leftrightarrow \left(\tfrac{1}{z}, \tfrac{1}{w}\right)$$

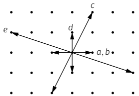


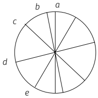
Spectral curve and isoradial graph







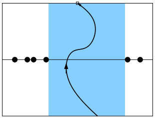




The integrable Ising model on isoradial graph (with Béatrice de Tilière et Kilian Raschel)

- elliptic weights. $k^2 \in (-\infty, 1)$: full range of coupling constant
- (k=0 correspond to critical case)
- For $k \neq 0$, the Kasteleyn operator on the Fisher graph has a unique inverse with bounded coefficients $K^{-1}_{x,y}$.
- These coefficients have a local expression

$$K^{-1}_{x,y} = \frac{k'}{8\pi} \int_{\Gamma_{x,y}} f_x(u+2K) f_y(u)\, \mathsf{Exp}_{\mathsf{x},\mathsf{y}}(u|k)\, du$$



- This operator can be used to define a Gibbs measure on spin contours (on the whole plane: Pfaffian process)
- Strong form of duality: $k \leftrightarrow k^*$ s.t. $(k^*)' \times k' = 1$
- Phase transition: same as for rooted spanning forests

Phase transition in the Ising model

Perspectives

- near critical scaling limit and massive free field (W.I.P.)
- Other models of statistical mechanics
  - Dimer model on bipartite isoradial graph
  - Paul Melotti: free-fermionic 8V-model
- Other 3D constistent equations on isoradial graphs
  - Adler, Bobenko, Suris: classification of such equations with (too many?) symmetries
- Integrable Ising model outside of the isoradial world
  - Marcin Lis: kite graphs
  - Dmitry Chelkak: S-embedding for Ising model