Geometric realizations of accordion complexes

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joint work with **Vincent Pilaud** (CNRS & LIX, Polytechnique)

JCB 2107, Bordeaux (LaBRI)
January 27$^{\text{th}}$, 2017

"Reminder": simplicial complexes

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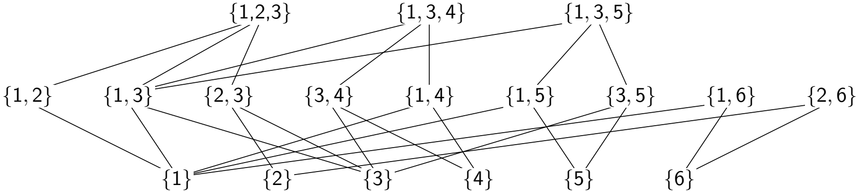
Example:

$$\Delta = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}$$
$$\{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,6\}, \{3,4\},$$
$$\{3,5\}, \{1,2,3\}, \{1,3,4\}, \{1,3,5\}\big\}$$



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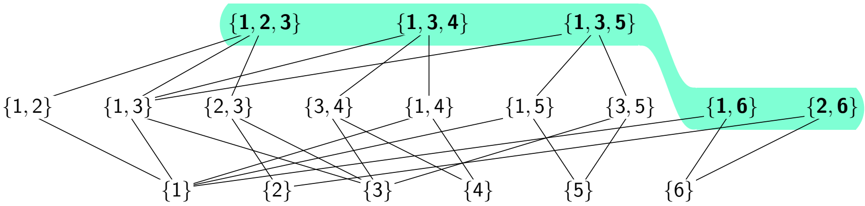

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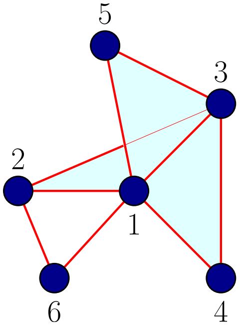

$=$ **(topological) Realization**

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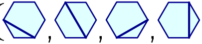
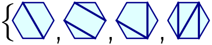
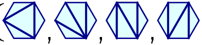
Example: *n*-simplicial associahedron

Vertices: **diagonals** $\{$  $,\ldots\}$

Faces: **dissections** $\{$  $,\ldots\}$

Facets: **triangulations** $\{$  $,\ldots\}$

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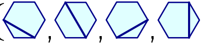
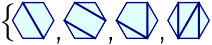
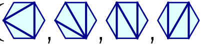
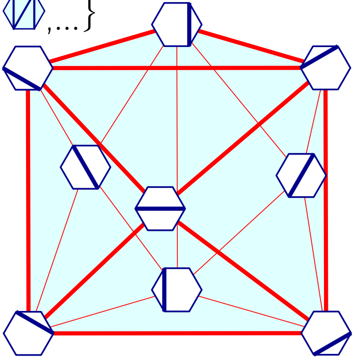
Vertices: **diagonals** $\{$  $, \dots \}$

Faces: **dissections** $\{$  $, \dots \}$

Facets: **triangulations** $\{$  $, \dots \}$



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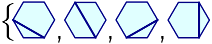
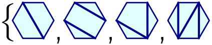
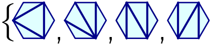
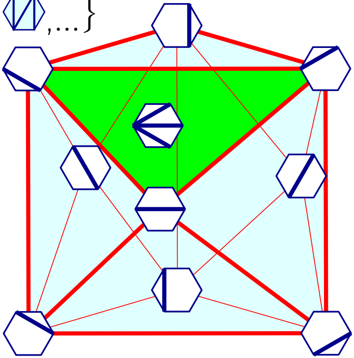
Vertices: **diagonals** $\{$$, \ldots\}$

Faces: **dissections** $\{$$, \ldots\}$

Facets: **triangulations** $\{$$, \ldots\}$



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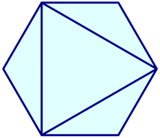
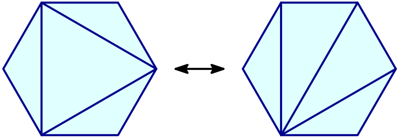
Flip graph of the $(n+3)$-gon

vertices: 

edges: 

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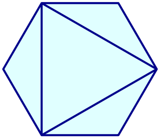

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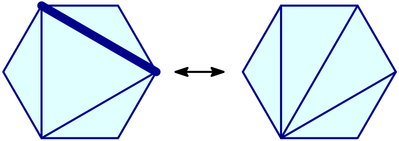

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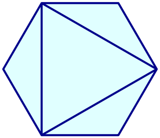
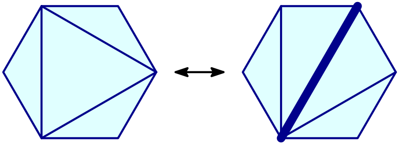
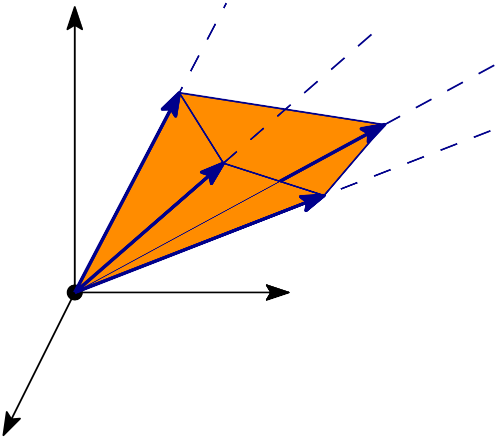
vertices: 

edges: 

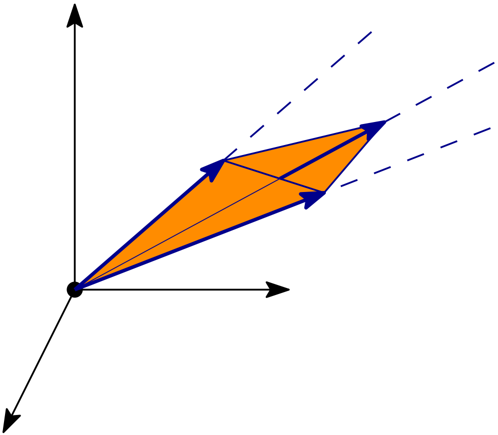
"Reminder": fans

Cone = positive span of a finite set of vectors



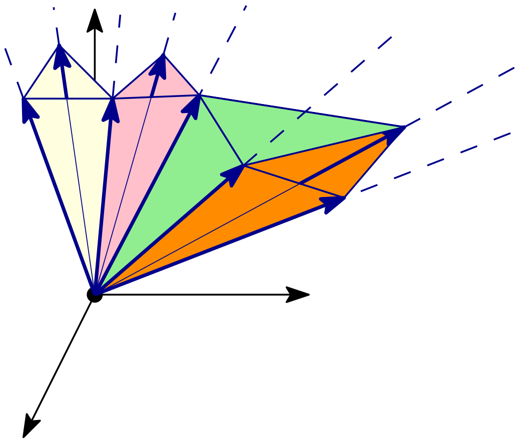
“Reminder”: fans

Simplicial cone = positive span of independent vectors



“Reminder”: fans

Fan = set of cones intersecting along faces



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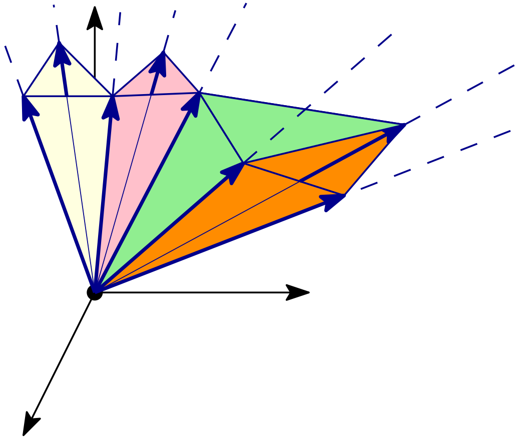
Simplicial fan = fan with simplicial cones



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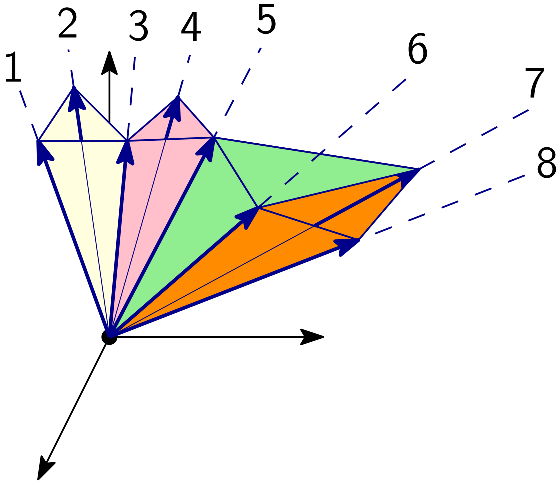

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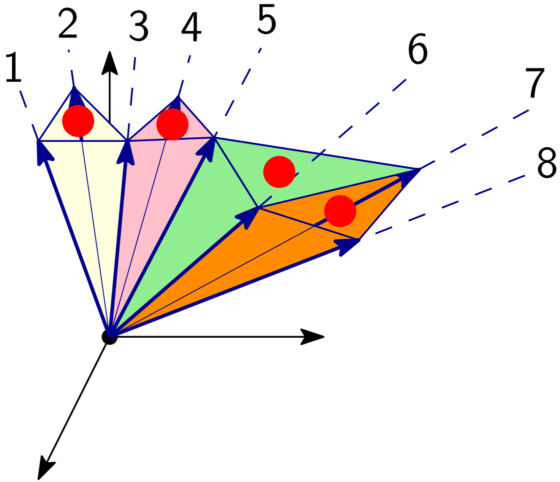

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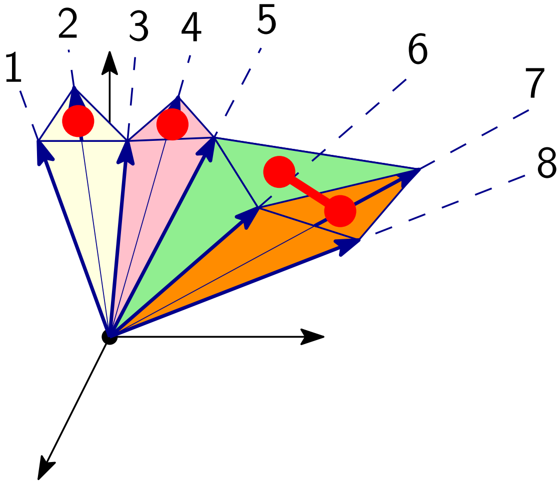

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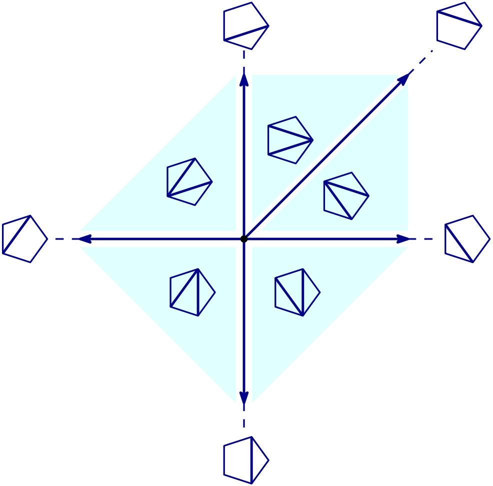
Complete fan = fan covering the whole space



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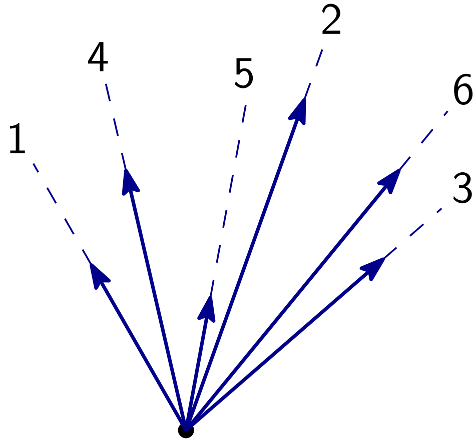
$$\Delta = \big\{\{1,2,3\},\{4,5,6\}\big\}$$



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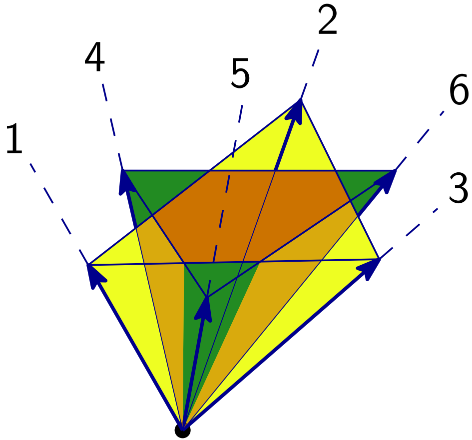
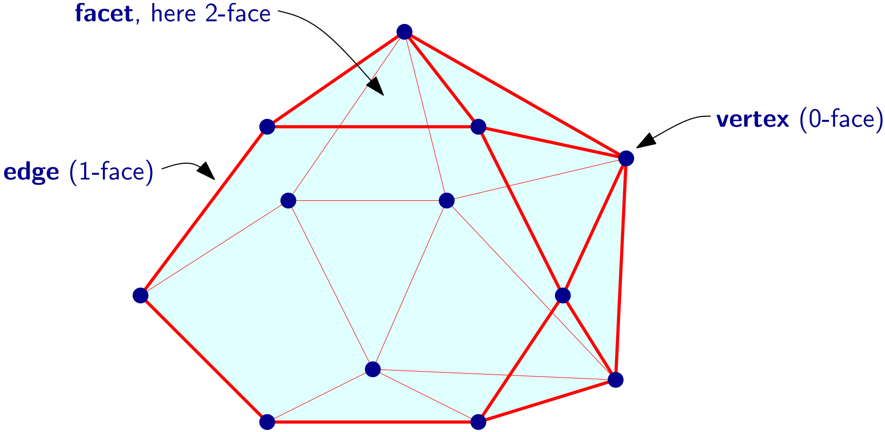


"Reminder":polytopes

Polytope = convex hull of finitely many points in $\mathbb{R}^n$



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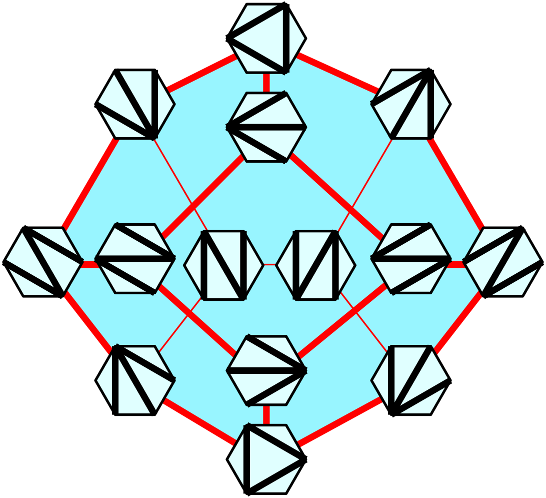
Simple polytope = polytope P with $\dim(P)$-regular graph



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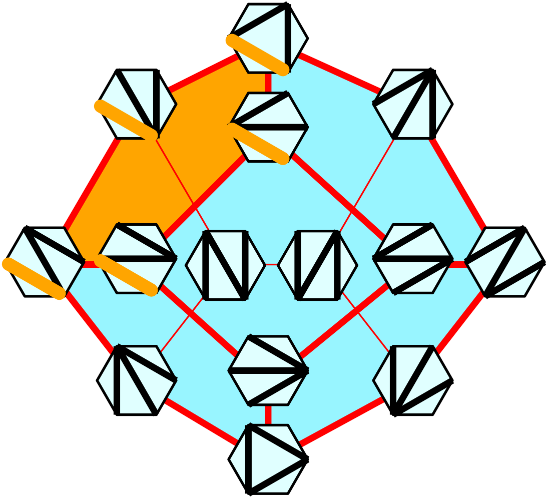

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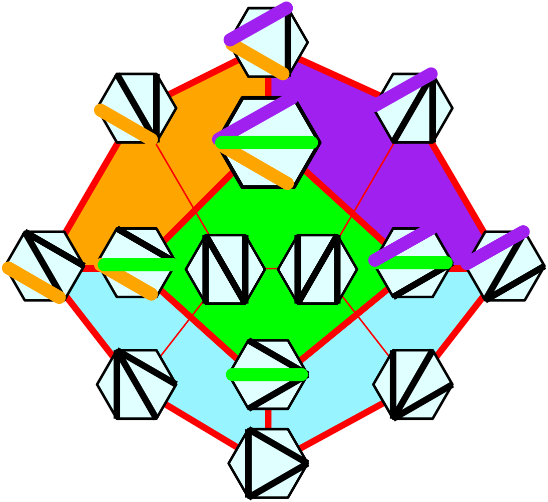

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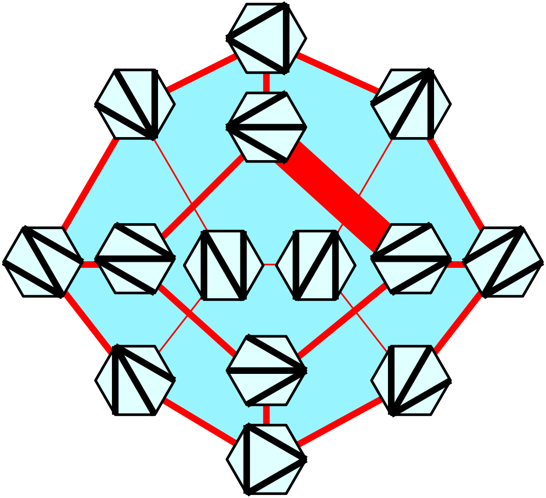

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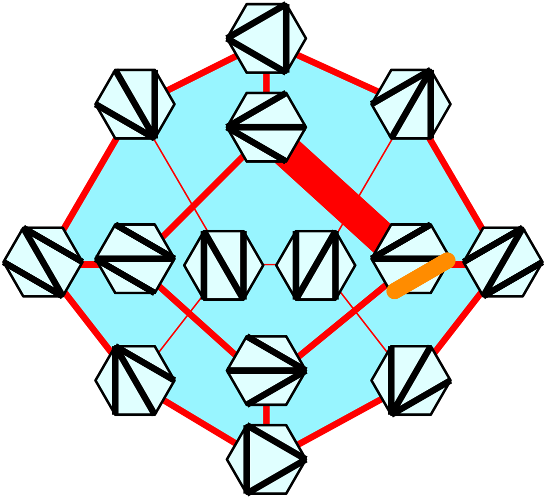

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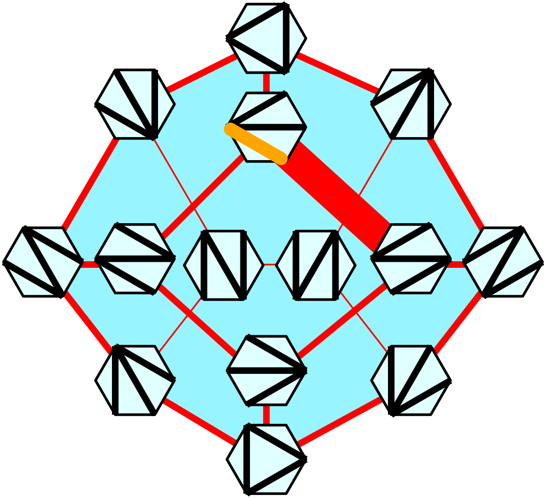

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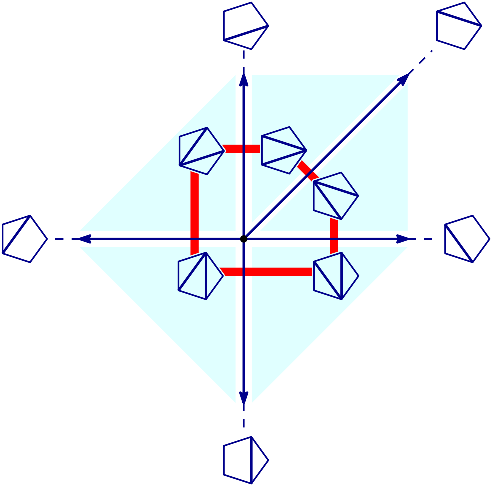
Normal fan = fan generated by outgoing normals to facets



History

Baryshnikov, *On Stokes sets* (2001)

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Chapoton, *Stokes posets and serpent nests* (2016)

Are Stokes posets lattices?

Are Stokes complexes realizable as polytopes?

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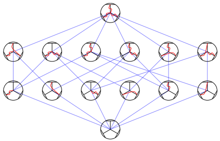

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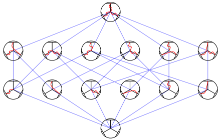

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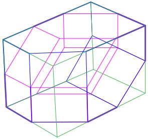

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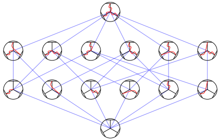

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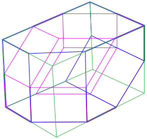

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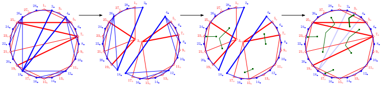
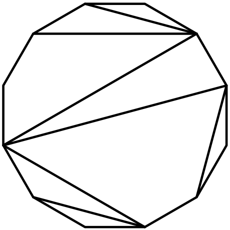
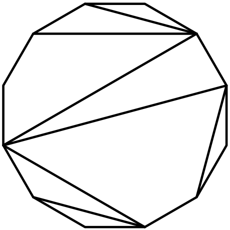


Accordion complexes



dissection

Accordion complexes



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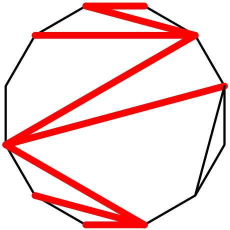
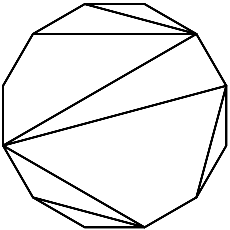
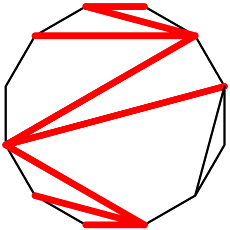
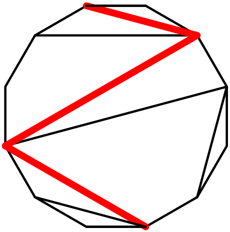


accordion

Accordion complexes







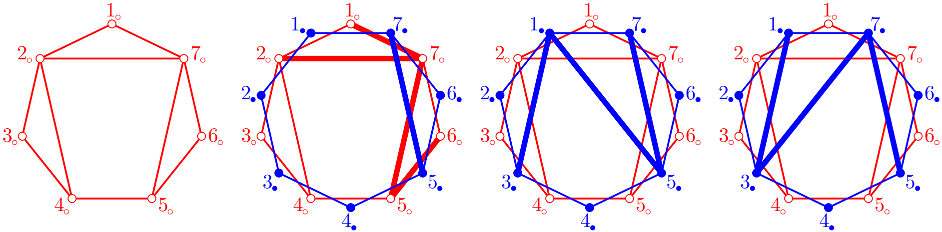
dissection

accordion

zigzag

Accordion complexes

Accordion complex of $\mathrm{D}_\circ = \mathcal{AC}(\mathrm{D}_\circ) := \{\mathrm{D}_\circ\text{-accordion dissections}\}$



$\mathrm{D}_\circ$
reference dissection

$\mathrm{D}_\circ$-**accordion diagonal**

$\mathrm{D}_\circ$-**accordion dissections**

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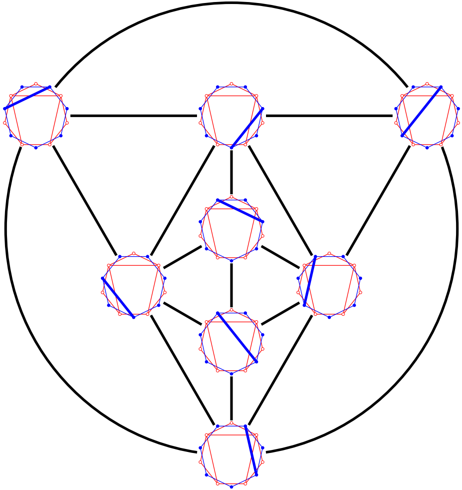
Example:



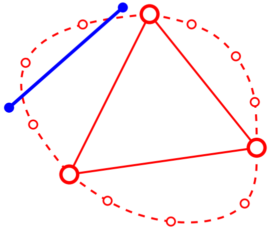
Accordion complexes

Accordion complex of $\mathrm{D}_\circ$ = $\mathcal{AC}(\mathrm{D}_\circ) := \{\mathrm{D}_\circ\text{-accordion dissections}\}$

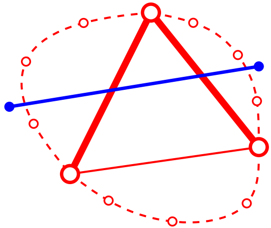
Example:

$\mathrm{D}_\circ$ is a triangulation $\Longrightarrow$ $\mathcal{AC}(\mathrm{D}_\circ)$ is the associahedron

Proof. *Any two edges of a triangle are incident.* ∎



or



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Proposition (Garver–McConville 2016)

The complex $\mathcal{AC}(\mathrm{D}_\circ)$ is a ***pseudo-manifold****:*

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*The complex $\mathcal{AC}(\mathrm{D}_\circ)$ is a **pseudo-manifold**:*
***Pure**: $\mathrm{D}_\bullet$ maximal $\mathrm{D}_\circ$-accordion dissection $\implies |\mathrm{D}_\bullet| = |\mathrm{D}_\circ|$.*

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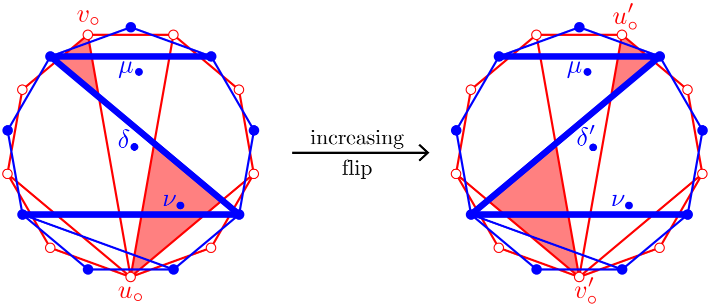
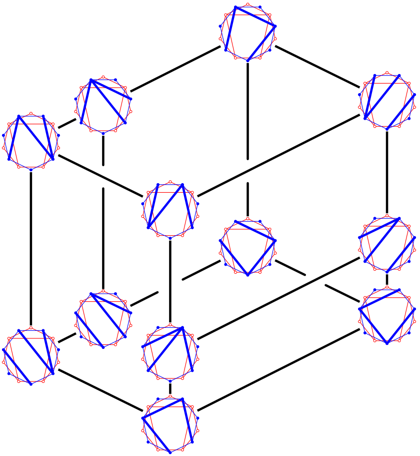
Pure*: $\mathrm{D}_\bullet$ maximal $\mathrm{D}_\circ$-accordion dissection $\implies |\mathrm{D}_\bullet| = |\mathrm{D}_\circ|$.*
Thin*: Flip operation on maximal $\mathrm{D}_\circ$-accordion dissections.*



Accordion complexes

Proposition (Garver–McConville 2016)

The oriented flip graph of $\mathrm{D}_\circ$ is the Hasse diagram of a lattice.



Accordion complexes

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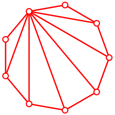
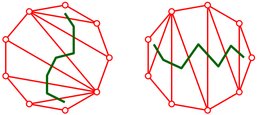
Examples:



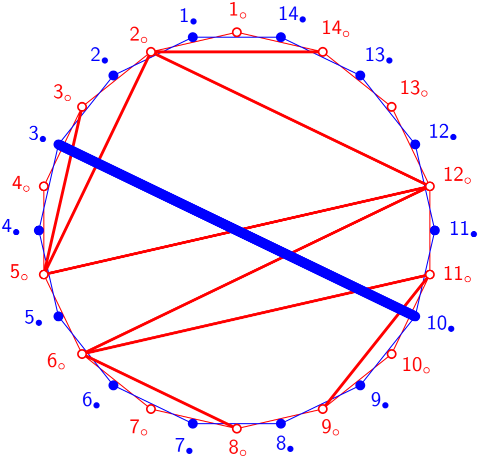
Tamari, *Monoïdes préodronnés et chaînes de Malcev* (1951)



Reading, *Cambrian lattices* (2006)

Geometric realization of accordion complexes I

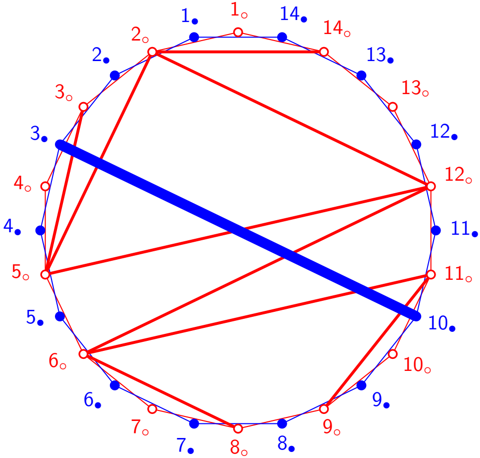
$\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal $\longrightarrow$ vector in $\mathbb{R}^{\mathrm{D}_\circ}$?



Geometric realization of accordion complexes I

$\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal

$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) :=$ alternating ± 1 on the zigzag of $\delta_\bullet$



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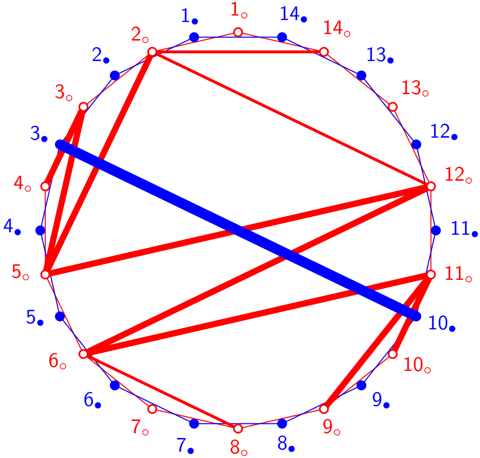

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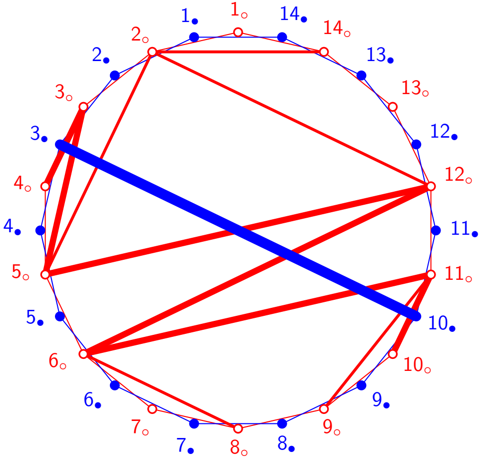

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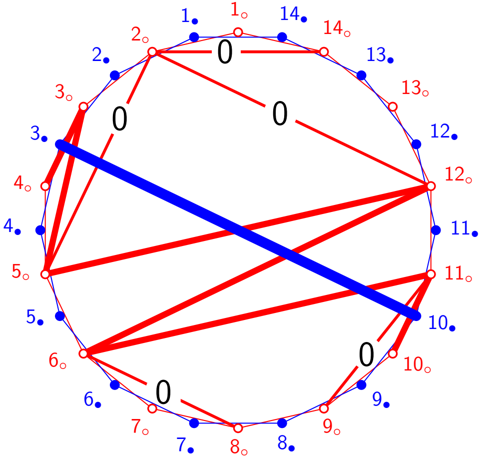
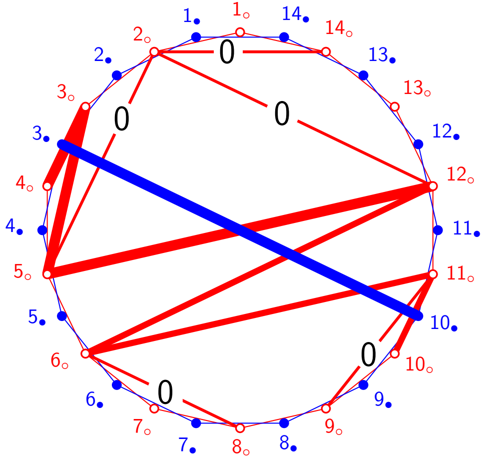


Geometric realization of accordion complexes I

$\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal

$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) :=$ alternating ± 1 on the zigzag of $\delta_\bullet$ with $\begin{cases} +1\text{'s on Z's} \\ -1\text{'s on } \Sigma\text{'s} \end{cases}$



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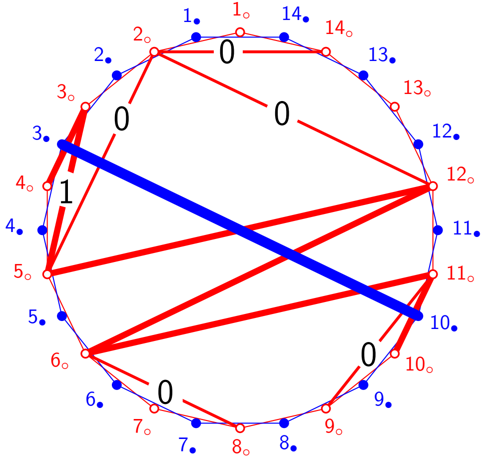

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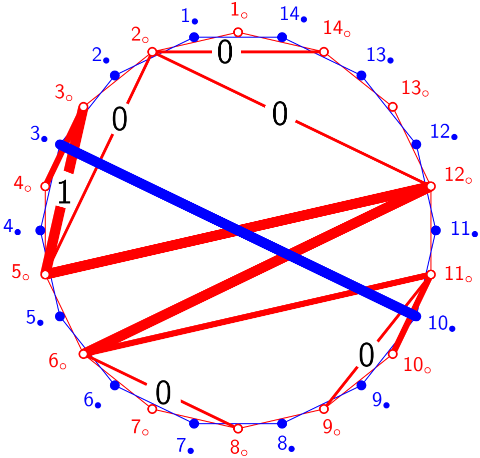

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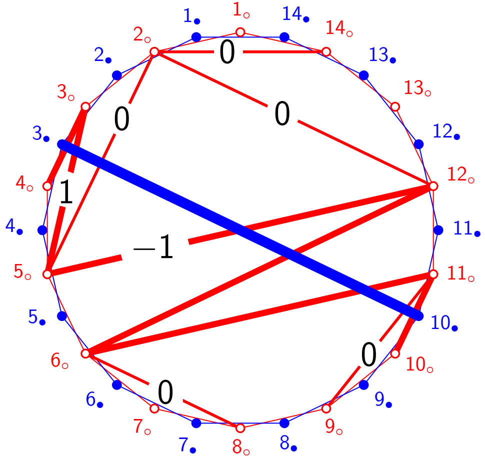

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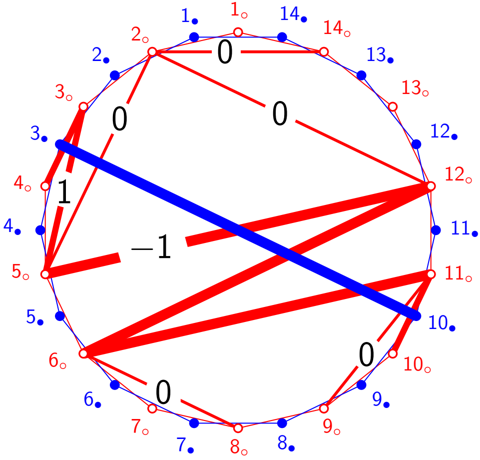

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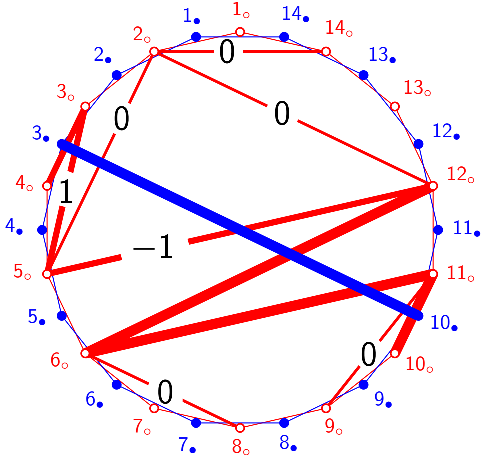

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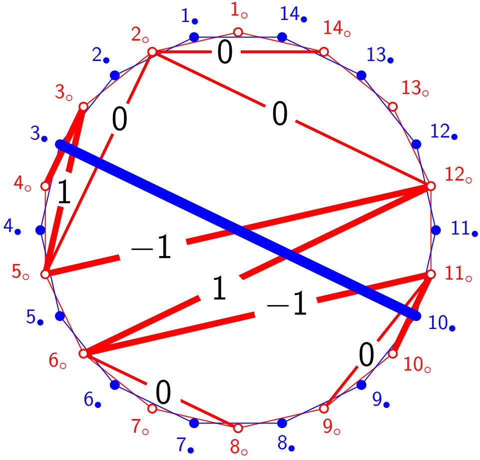
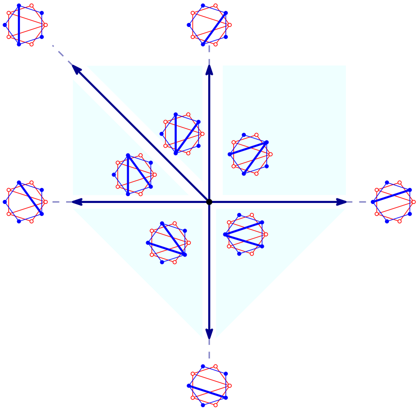


Geometric realization of accordion complexes I

Theorem (M.–Pilaud 2017$^+$)

The $\mathbf{g}$*-vectors of* $\mathrm{D}_\circ$*-accordion diagonals are the rays of a complete simplicial fan* $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ *realizing* $\mathcal{AC}(\mathrm{D}_\circ)$.



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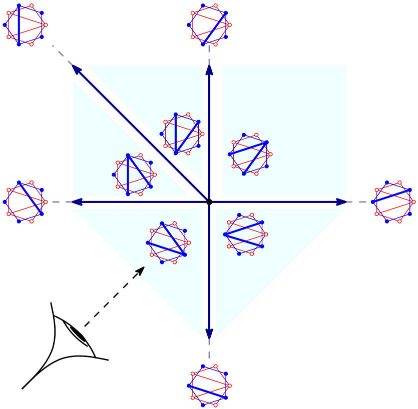

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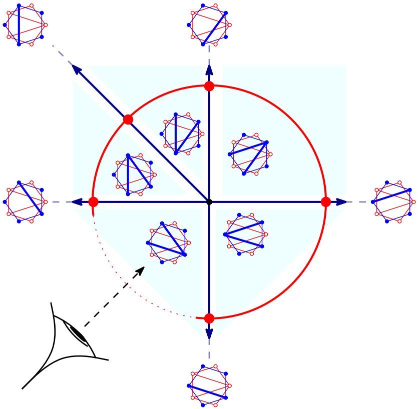

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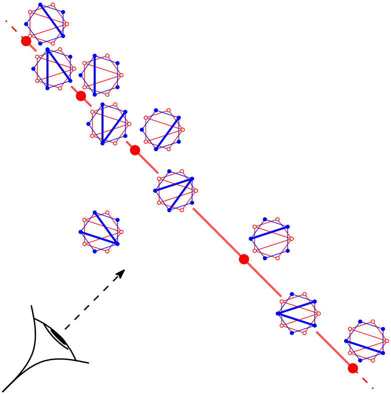

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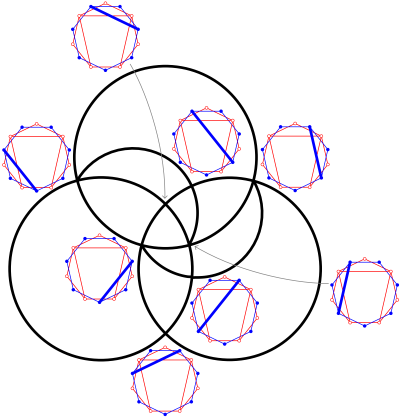

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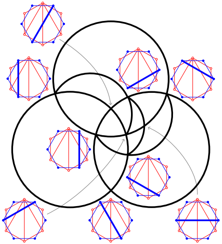
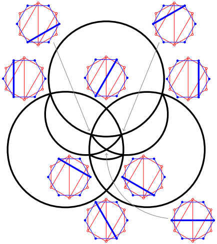
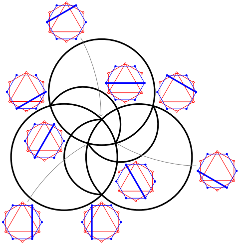


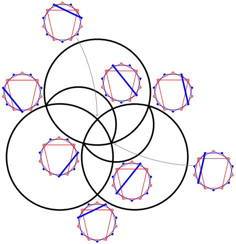
Loday



Hohlweg–Lange
Reading



Hohlweg–Pilaud–Stella



M.–Pilaud

Geometric realization of accordion complexes I

Theorem (M.–Pilaud 2017$^+$)

The $\mathbf{g}$*-vector fan* $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ *is the normal fan of a polytope* $\mathsf{Acco}(\mathrm{D}_\circ)$.

Geometric realization of accordion complexes I

Theorem (M.–Pilaud 2017$^+$)

The $\mathbf{g}$*-vector fan* $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ *is the normal fan of a polytope* $\mathrm{Acco}(\mathrm{D}_\circ)$.

Step 1: $\{\mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet) \,|\, \delta_\bullet \in \mathrm{D}_\bullet\}$:= dual basis of $\{\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) \,|\, \delta_\bullet \in \mathrm{D}_\bullet\}$

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Step 2:

$$\mathrm{Zono}(\mathrm{D}_\circ) := \sum_{\substack{\mathrm{D}_\bullet \in \mathcal{AC}(\mathrm{D}_\circ),\, |\mathrm{D}_\bullet| = |\mathrm{D}_\circ| \\ \delta_\bullet \in \mathrm{D}_\bullet}} \mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet)$$

Geometric realization of accordion complexes I

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$$\mathbf{A} + \mathbf{B} := \{\mathsf{a} + \mathsf{b} \,|\, \mathsf{a} \in \mathbf{A}, \mathsf{b} \in \mathbf{B}\}$$

Geometric realization of accordion complexes I

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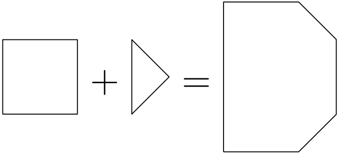
$$\mathbf{A} + \mathbf{B} := \{a + b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$$



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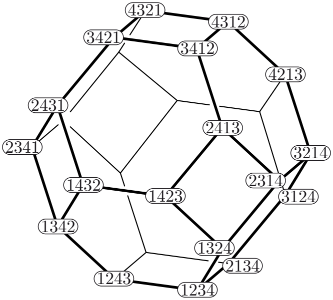

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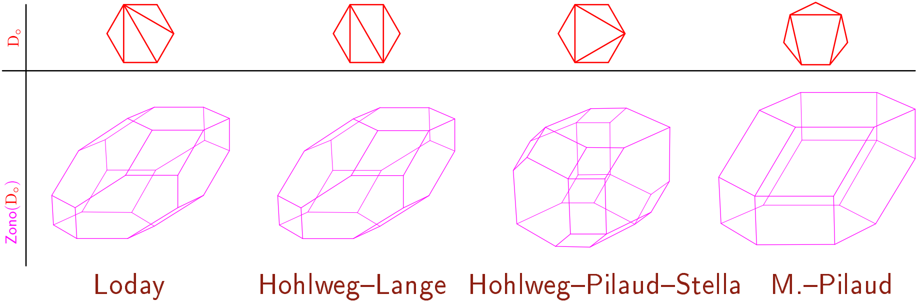

Geometric realization of accordion complexes I

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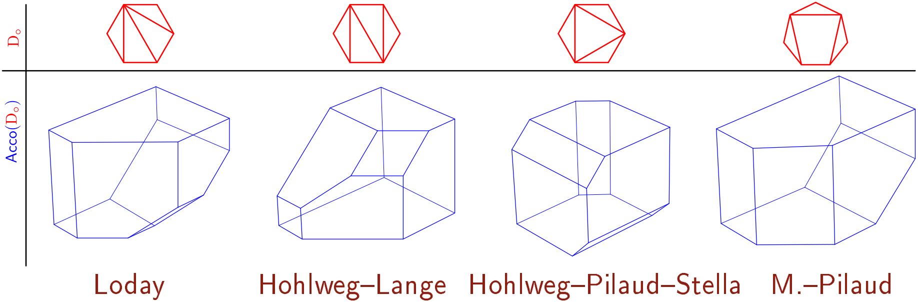
Step 3: keep only inequalities perpendicular to $\mathbf{g}$-vectors



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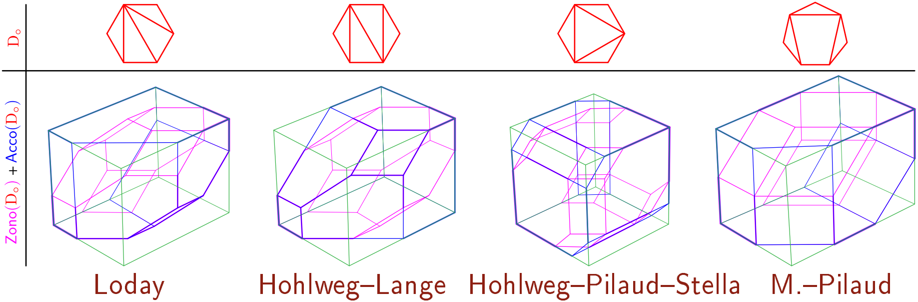
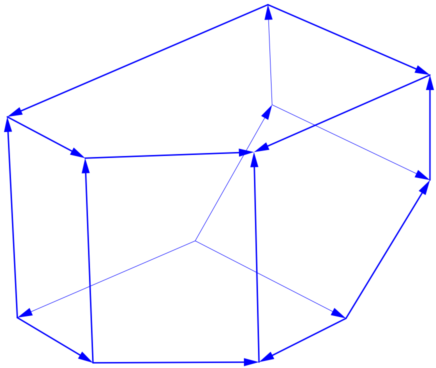


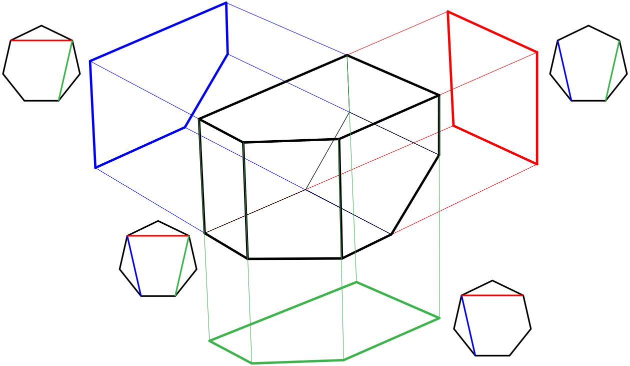
Geometric realization of accordion complexes I

Orientation



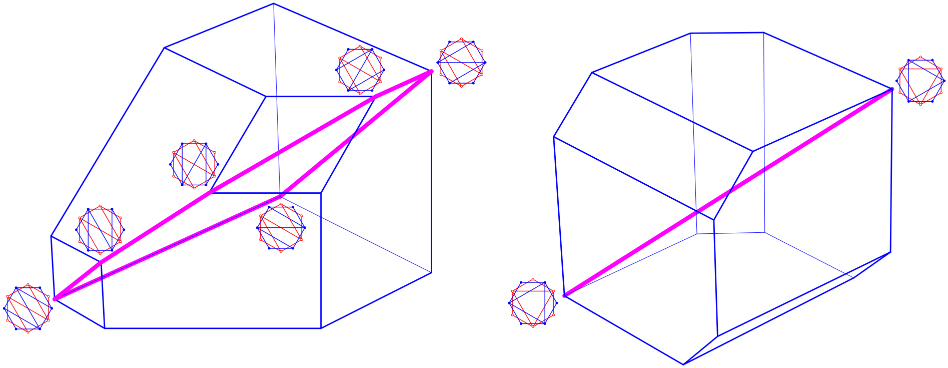
Geometric realization of accordion complexes I

Projection

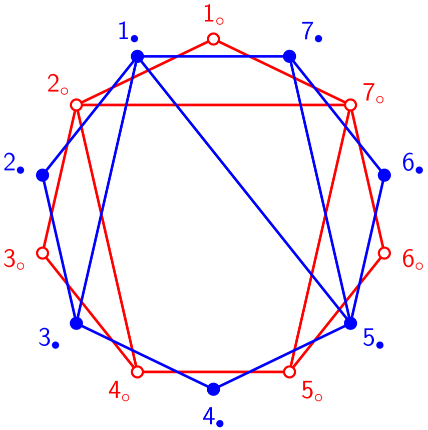


Geometric realization of accordion complexes I

Symmetries

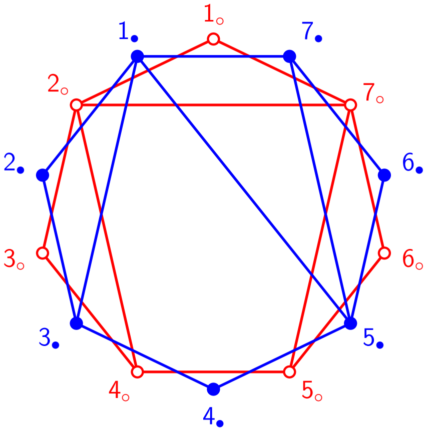


Geometric realization of accordion complexes I



c-vectors?

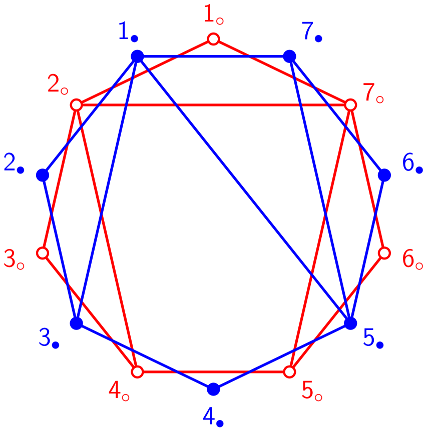
Geometric realization of accordion complexes I



$$\begin{array}{c} \\ 2_\circ 4_\circ \\ 2_\circ 7_\circ \\ 5_\circ 7_\circ \end{array} \begin{array}{c} \begin{array}{ccc} 1_\bullet 3_\bullet & 1_\bullet 5_\bullet & 5_\bullet 7_\bullet \end{array} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

g-vectors

Geometric realization of accordion complexes I



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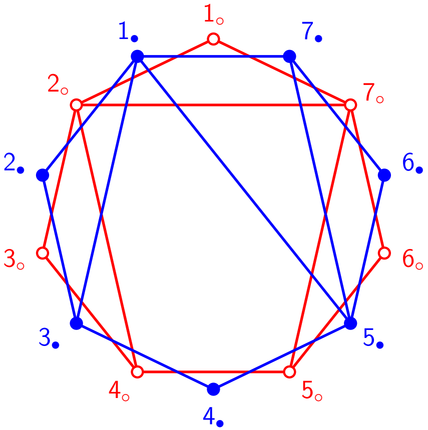
$\mathbf{g}$-vectors

Geometric realization of accordion complexes I



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c-vectors

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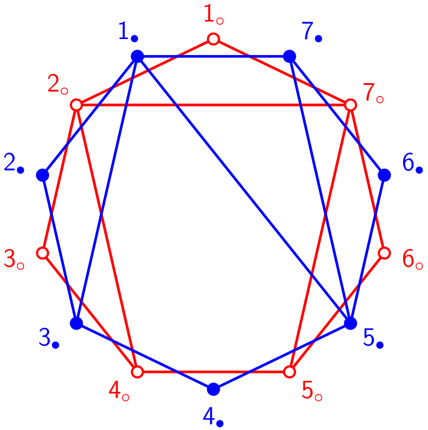
Proposition (Reciprocity, M.–Pilaud 2017$^+$)

$D_\bullet$ *max.* $D_\circ$*-accordion diss.* $\iff$ $D_\circ$ *max.* $D_\bullet$*-accordion diss.*



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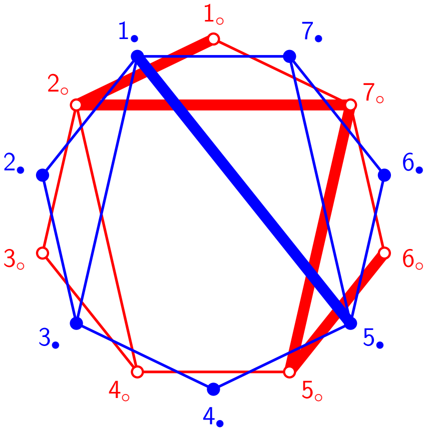


$$\begin{array}{c|ccc} & 1_\bullet 3_\bullet & 1_\bullet 5_\bullet & 5_\bullet 7_\bullet \\ \hline 2_\circ 4_\circ & -1 & 0 & 0 \\ 2_\circ 7_\circ & 0 & 1 & 0 \\ 5_\circ 7_\circ & 0 & -1 & 1 \end{array}$$

g-vectors

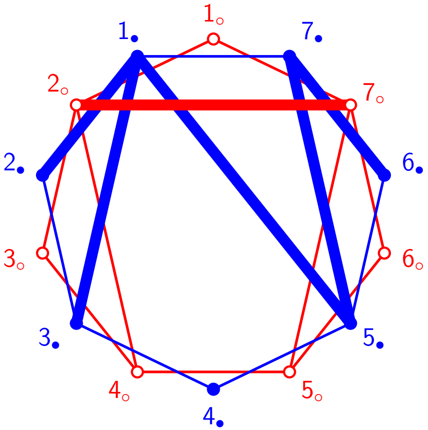
$$\begin{array}{c|ccc} & 2_\circ 4_\circ & 2_\circ 7_\circ & 5_\circ 7_\circ \\ \hline 1_\bullet 3_\bullet & -1 & 0 & 0 \\ 1_\bullet 5_\bullet & 0 & 1 & 0 \\ 5_\bullet 7_\bullet & 0 & -1 & 1 \end{array}$$

c-vectors

Geometric realization of accordion complexes I

Proposition (Reciprocity, M.–Pilaud 2017^{+})

$D_\bullet$ *max.* $D_\circ$*-accordion diss.* $\Longleftrightarrow$ $D_\circ$ *max.* $D_\bullet$*-accordion diss.*



$$\begin{array}{c|ccc} & 1_\bullet 3_\bullet & 1_\bullet 5_\bullet & 5_\bullet 7_\bullet \\ \hline 2_\circ 4_\circ & -1 & 0 & 0 \\ 2_\circ 7_\circ & 0 & 1 & 0 \\ 5_\circ 7_\circ & 0 & -1 & 1 \end{array}$$

g-vectors

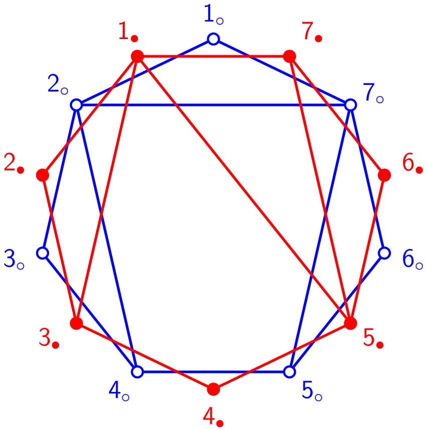
$$\begin{array}{c|ccc} & 2_\circ 4_\circ & 2_\circ 7_\circ & 5_\circ 7_\circ \\ \hline 1_\bullet 3_\bullet & -1 & 0 & 0 \\ 1_\bullet 5_\bullet & 0 & 1 & 0 \\ 5_\bullet 7_\bullet & 0 & -1 & 1 \end{array}$$

c-vectors

Geometric realization of accordion complexes I

Proposition (Reciprocity, M.–Pilaud 2017+)

$\color{blue}{D_\bullet}$ *max.* $\color{red}{D_\circ}$*-accordion diss.* $\iff$ $\color{red}{D_\circ}$ *max.* $\color{blue}{D_\bullet}$*-accordion diss.*



$$\begin{array}{c|ccc} & 1_\bullet 3_\bullet & 1_\bullet 5_\bullet & 5_\bullet 7_\bullet \\ \hline 2_\circ 4_\circ & -1 & 0 & 0 \\ 2_\circ 7_\circ & 0 & 1 & 0 \\ 5_\circ 7_\circ & 0 & -1 & 1 \end{array}$$

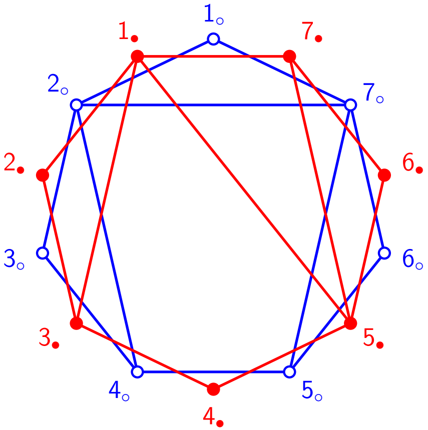
$$\begin{array}{c|ccc} & 2_\circ 4_\circ & 2_\circ 7_\circ & 5_\circ 7_\circ \\ \hline 1_\bullet 3_\bullet & -1 & 0 & 0 \\ 1_\bullet 5_\bullet & 0 & 1 & 0 \\ 5_\bullet 7_\bullet & 0 & -1 & 1 \end{array}$$

c-vectors

Geometric realization of accordion complexes I

Proposition (Reciprocity, M.–Pilaud 2017$^+$)

$\mathrm{D}_\bullet$ *max.* $\mathrm{D}_\circ$*-accordion diss.* $\iff$ $\mathrm{D}_\circ$ *max.* $\mathrm{D}_\bullet$*-accordion diss.*



$$\begin{array}{c} \\ 2_\circ 4_\circ \\ 2_\circ 7_\circ \\ 5_\circ 7_\circ \end{array} \begin{array}{c} \begin{array}{ccc} 1_\bullet 3_\bullet & 1_\bullet 5_\bullet & 5_\bullet 7_\bullet \end{array} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ 1_\bullet 3_\bullet \\ 1_\bullet 5_\bullet \\ 5_\bullet 7_\bullet \end{array} \begin{array}{c} \begin{array}{ccc} 2_\circ 4_\circ & 2_\circ 7_\circ & 5_\circ 7_\circ \end{array} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ 1_\bullet 3_\bullet \\ 1_\bullet 5_\bullet \\ 5_\bullet 7_\bullet \end{array} \begin{array}{c} \begin{array}{ccc} 2_\circ 4_\circ & 2_\circ 7_\circ & 5_\circ 7_\circ \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \end{array}$$

c-vectors

Geometric realization of accordion complexes I

Proposition (Reciprocity, M.–Pilaud 2017$^+$)

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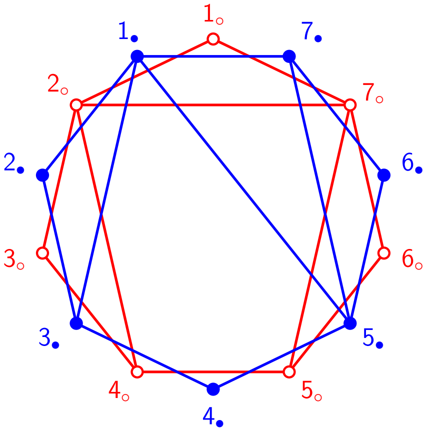

$$\begin{array}{c} \\ 2_\circ 4_\circ \\ 2_\circ 7_\circ \\ 5_\circ 7_\circ \end{array} \begin{array}{c} \begin{array}{ccc} 1_\bullet 3_\bullet & 1_\bullet 5_\bullet & 5_\bullet 7_\bullet \end{array} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ 1_\bullet 3_\bullet \\ 1_\bullet 5_\bullet \\ 5_\bullet 7_\bullet \end{array} \begin{array}{c} \begin{array}{ccc} 2_\circ 4_\circ & 2_\circ 7_\circ & 5_\circ 7_\circ \end{array} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ 1_\bullet 3_\bullet \\ 1_\bullet 5_\bullet \\ 5_\bullet 7_\bullet \end{array} \begin{array}{c} \begin{array}{ccc} 2_\circ 4_\circ & 2_\circ 7_\circ & 5_\circ 7_\circ \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \end{array}$$

c-vectors

Geometric realization of accordion complexes I

Proposition (Reciprocity, M.–Pilaud 2017^{+})

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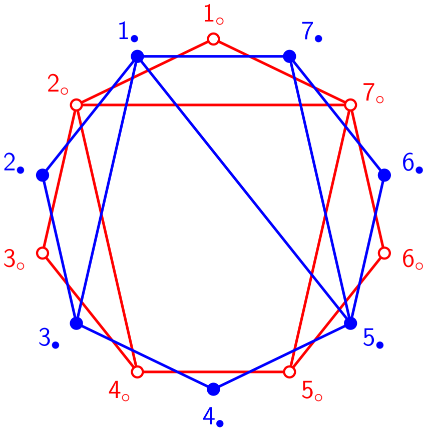
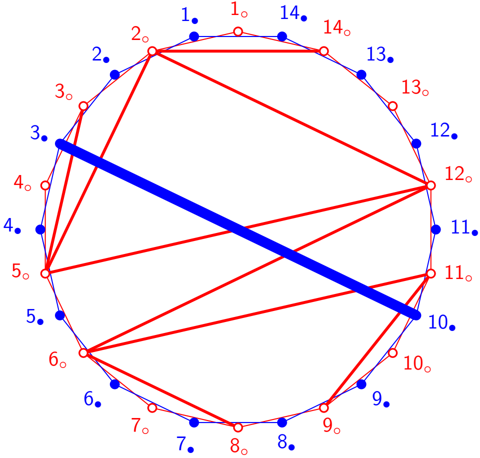


$$\begin{array}{c|c} & 1_\bullet3_\bullet \quad 1_\bullet5_\bullet \quad 5_\bullet7_\bullet \\ \begin{array}{c} 2_\circ4_\circ \\ 2_\circ7_\circ \\ 5_\circ7_\circ \end{array} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c|c} & 2_\circ4_\circ \quad 2_\circ7_\circ \quad 5_\circ7_\circ \\ \begin{array}{c} 1_\bullet3_\bullet \\ 1_\bullet5_\bullet \\ 5_\bullet7_\bullet \end{array} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}
=
\begin{array}{c|c} & 2_\circ4_\circ \quad 2_\circ7_\circ \quad 5_\circ7_\circ \\ \begin{array}{c} 1_\bullet3_\bullet \\ 1_\bullet5_\bullet \\ 5_\bullet7_\bullet \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \end{array} \times -1$$

Geometric realization of accordion complexes II

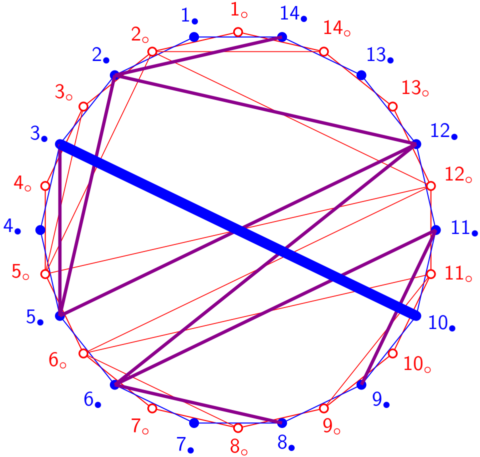
$\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal $\longrightarrow$ vector in $\mathbb{R}^{\mathrm{D}_\circ}$?



Geometric realization of accordion complexes II

$\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal

$\mathbf{d}(\mathrm{D}_\circ, \delta_\bullet) :=$ characteristic vector of crossings with the diagonals of $\rho(\mathrm{D}_\circ)$



Geometric realization of accordion complexes II

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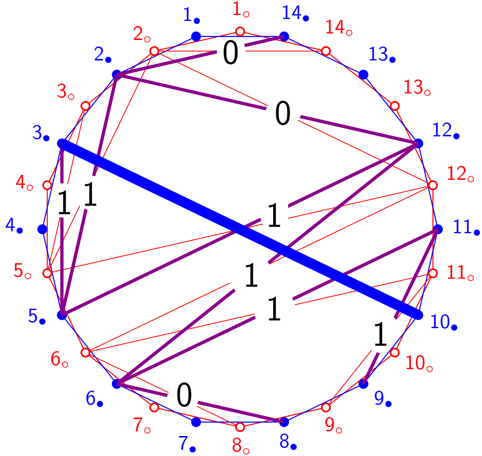

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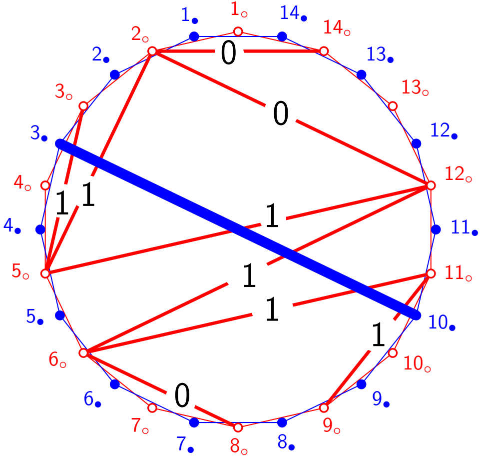

Geometric realization of accordion complexes II

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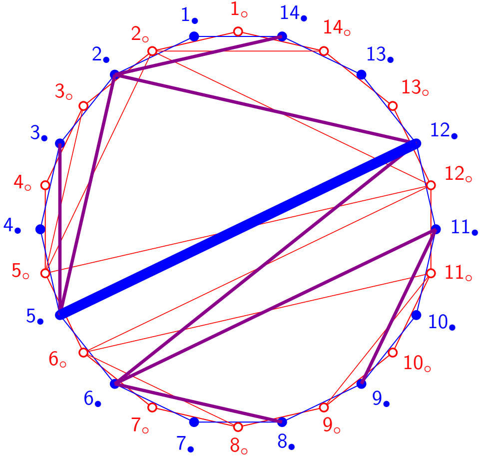
$\mathbf{d}(\mathrm{D}_\circ, \delta_\bullet) :=$ characteristic vector of crossings with the diagonals of $\rho(\mathrm{D}_\circ)$



Geometric realization of accordion complexes II

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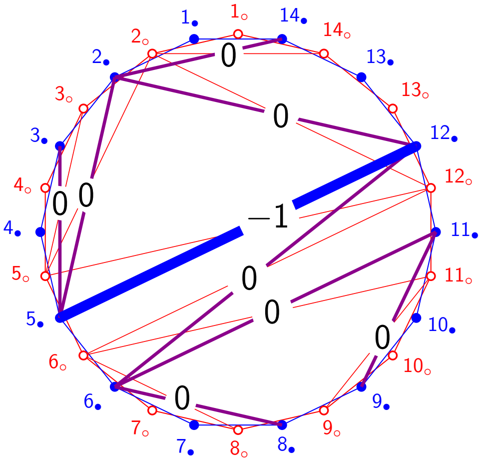
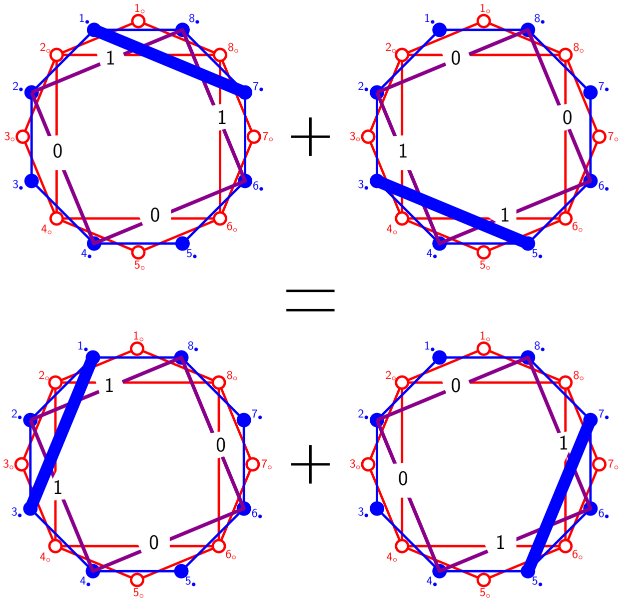


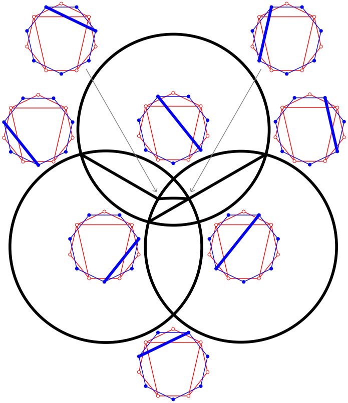
Geometric realization of accordion complexes II



Geometric realization of accordion complexes II

Theorem (M.–Pilaud 2017^{+})

The $\mathbf{d}$*-vectors of* $\mathrm{D}_\circ$*-accordion diagonals are the rays of a complete simplicial fan* $\mathcal{F}^{\mathbf{d}}(\mathrm{D}_\circ)$ *realizing* $\mathcal{AC}(\mathrm{D}_\circ)$ *if and only if* $\mathrm{D}_\circ$ *has no even interior cell.*



Geometric realization of accordion complexes II

Theorem (M.–Pilaud 2017$^+$)

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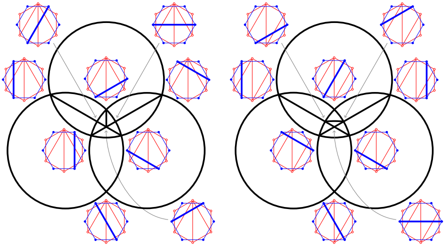
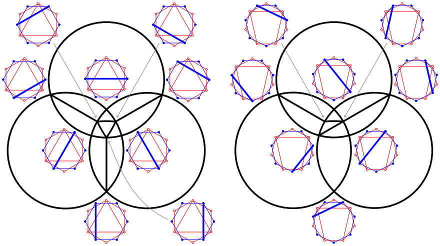
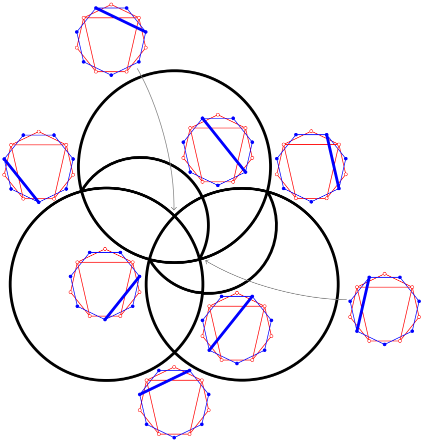


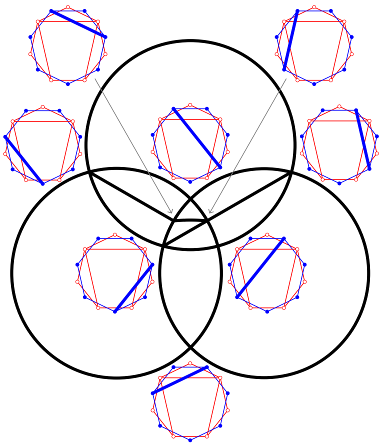


Fomin–Zelevinsky   Stella   Santos   M.–Pilaud

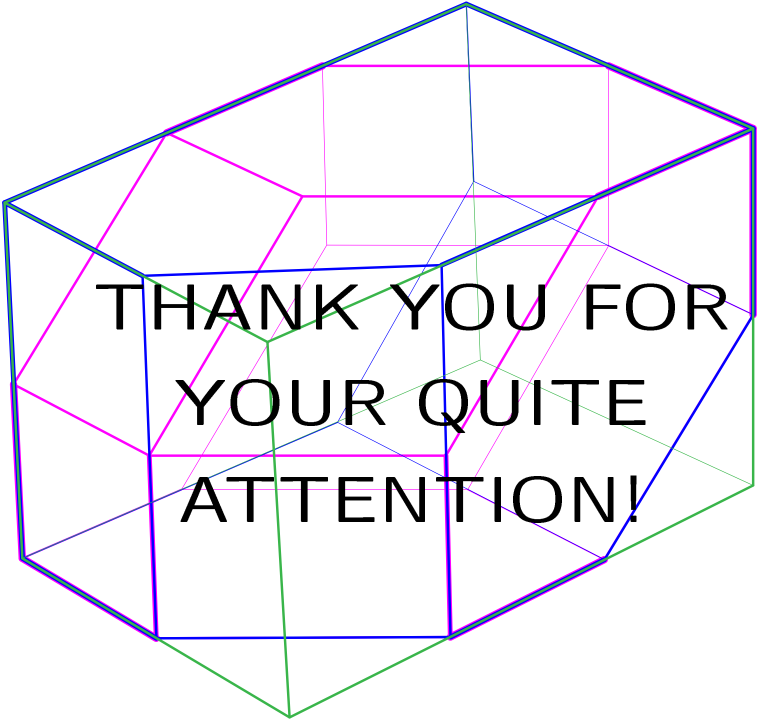
Sum up



polytopal **g**-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ for all dissections $\mathrm{D}_\circ$



d-vector fan $\mathcal{F}^{\mathbf{d}}(\mathrm{D}_\circ)$ for $\mathrm{D}_\circ$ without even interior cell



THANK YOU FOR YOUR QUITE ATTENTION!