Hamiltonian cycles in planar cubic graphs

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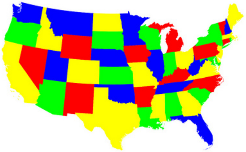
JCB, January 2017

Four Color Problem, mentioned first as early as 1840, states that given any separation of a plane into contigous regions, called a *map*, the regions can be colored using at most four colors so that no two adjacent regions have the same color.



The problem can be formulated using the language of graph theory: Each region of a map is respresented by a vertex of a graph, two vertices are joined by an edge for every pair of regions that share a boundary segment.

This graph is always *planar*, i.e., it can be drawn in the plane without crossings by placing each vertex at an arbitrarily chosen location within the regions it corresponds to, and by drawing the edges as curves joining the corresponding vertices.

The *dual* graph captures in fact the mutual position of the regions too – each edge of the dual graph is a boundary segment between two regions.

It is rare that more than three regions meet at the same point. Therefore, the dual graph is cubic (3-regular). It means that the original graph is a triangulation. If it was not, it could be transformed into one by adding edges.

Connection to 3-edge-colorings

Any 4-coloring of a planar triangulation induces a 3-edge-coloring the dual graph. Conversely, a 3-edge-coloring of a cubic planar graph induces a 4-coloring of the dual graph.

$$AB, CD \to 1$$
$$AC, BD \to 2$$
$$AD, BC \to 3$$

How to find a 3-edge-coloring of a cubic graph?

Conjecture (Tait 1884)

Every 3-connected cubic planar graph has a Hamilton cycle.

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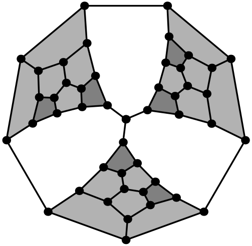
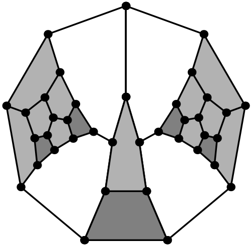
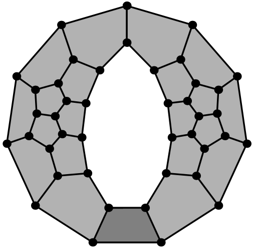


Smallest non-hamiltonian 3-connected cubic planar graph have 38 vertices (Bosák 1971; Holton and McKay 1988), there are 6 of them:



Smallest non-hamiltonian 3-connected cubic planar graph without cyclic 3-edge-cuts has 42 vertices (Faulkner and Younger 1974):



Conjecture (Barnette, 1969)

All 3-connected cubic planar graphs with faces of size at most 6 are hamiltonian.

Recall that $3f_3 + 2f_4 + f_5 = 12$ for all graphs from Barnette's conjecture.

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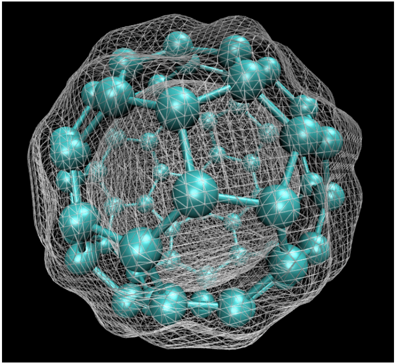
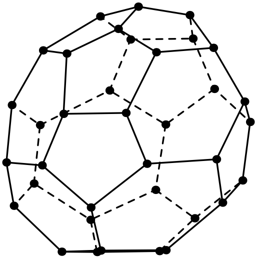
- *Fullerenes* are carbon molecules, where atoms are arranged in pentagons and hexagons. They can be represented by convex polytopes or by plane graphs.



- A *fullerene graph* is a 3-connected cubic plane graph with pentagonal and hexagonal faces.



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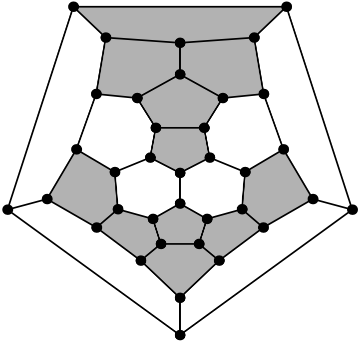
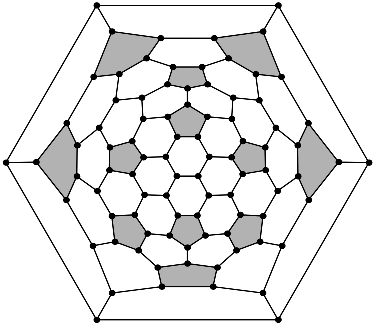


Let the length of the longest cycle in a fullerene graph on n vertices be denoted by $\Phi(n)$.

Jendrol', Owens (1995): $\Phi(n) \geq \frac{4}{5}n$.

Král', Pangrác, Sereni, Škrekovski (2007): $\Phi(n) \geq \frac{5}{6}n - \frac{2}{3}$.



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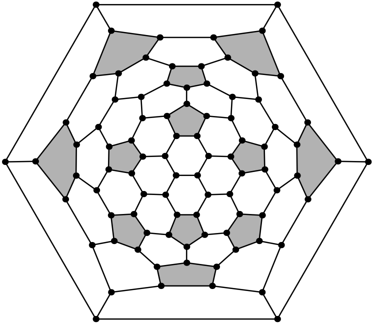

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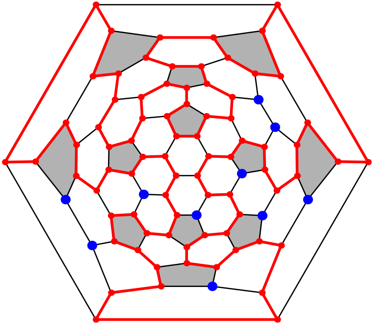

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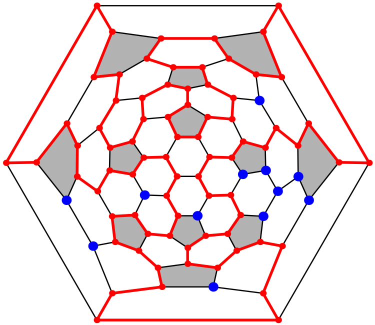

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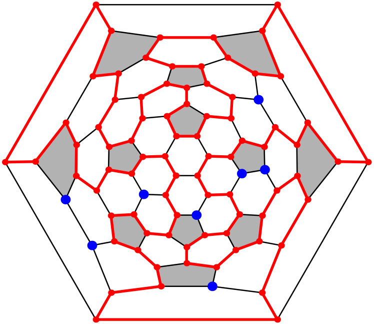

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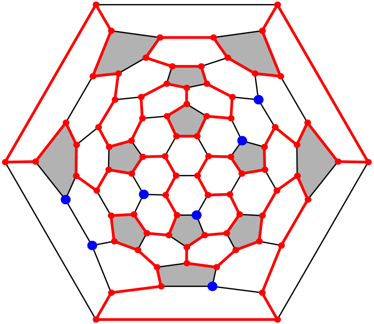

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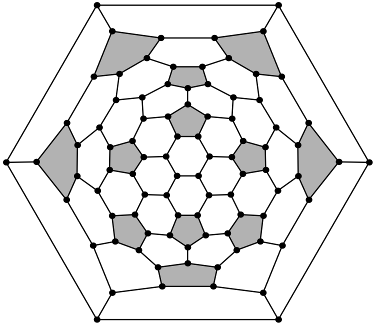

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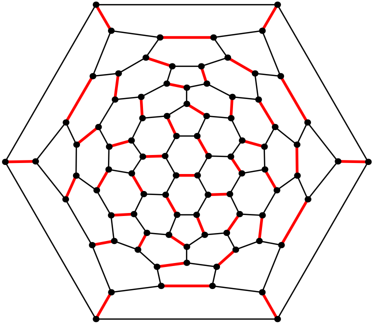

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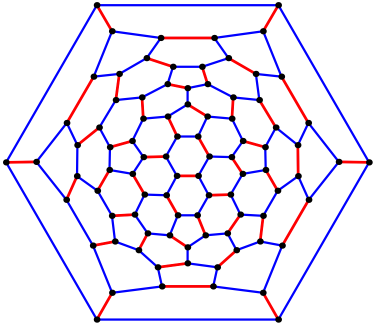

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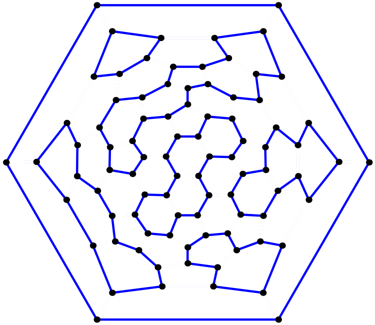

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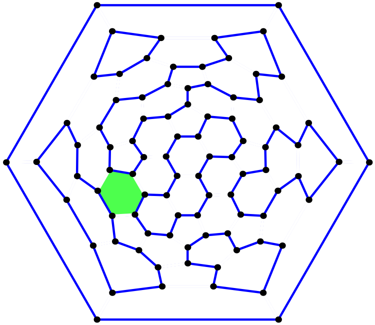

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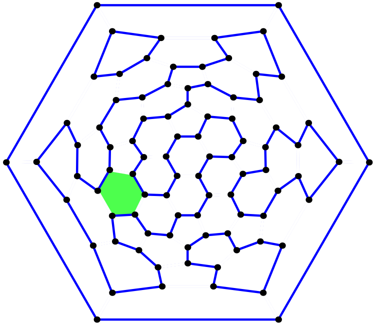

Long cycles in fullerene graphs : gluing small cycles together

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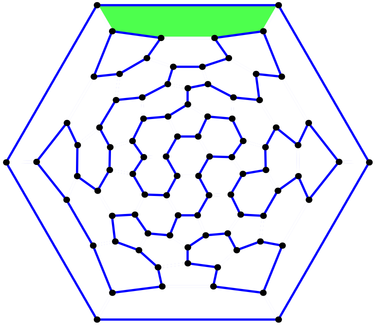

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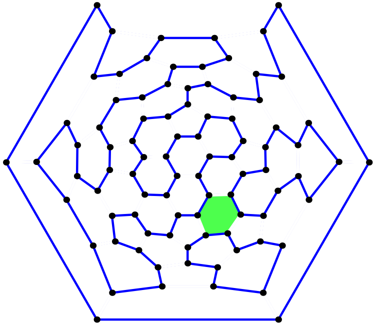

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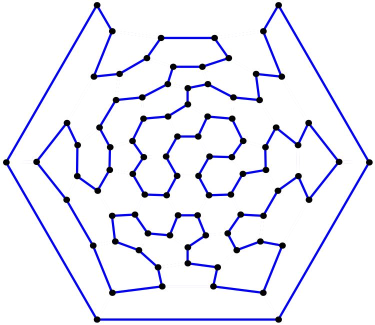
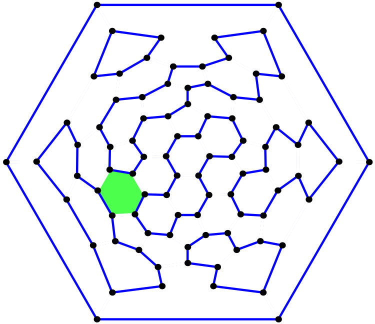


A hexagonal face f is called *resonant* with respect to a perfect matching M (or the corresponding 2-factor $G \setminus M$), if precisely three edges incident with f are in M (and three in $G \setminus M$).



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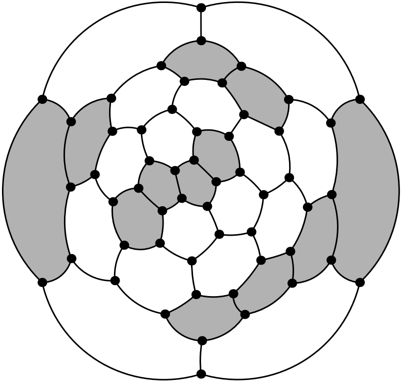

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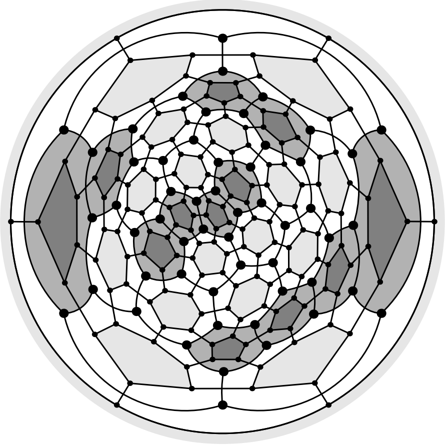

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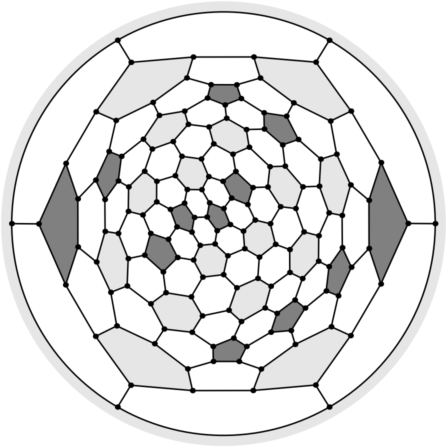

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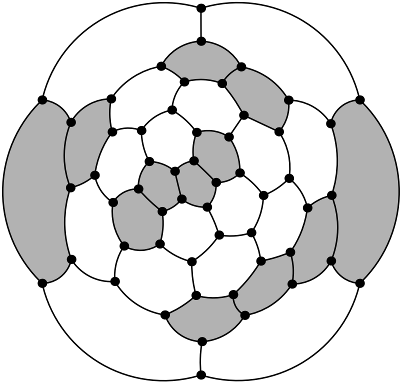

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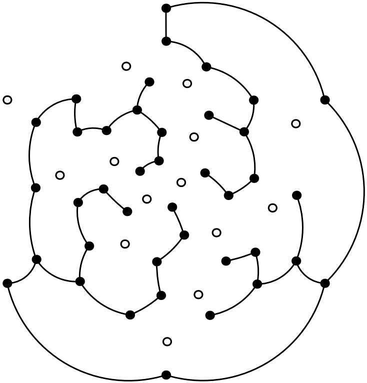

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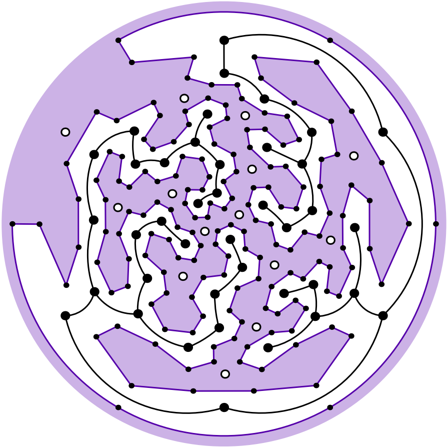

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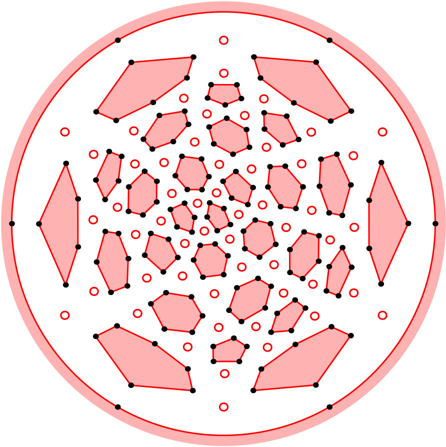

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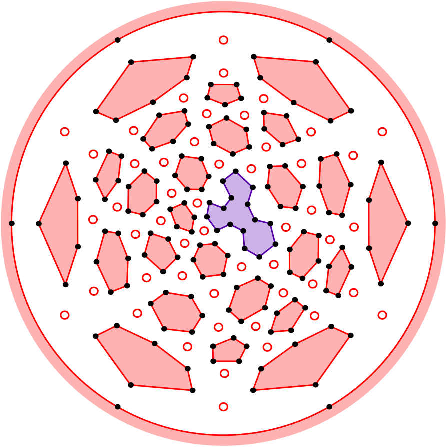

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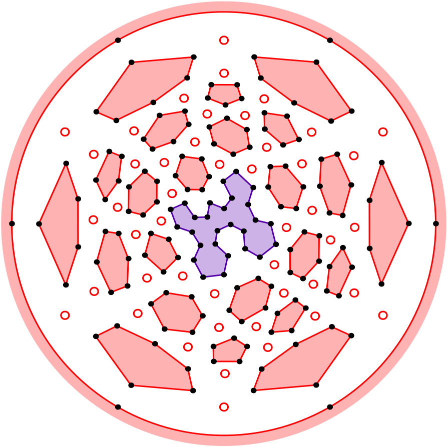

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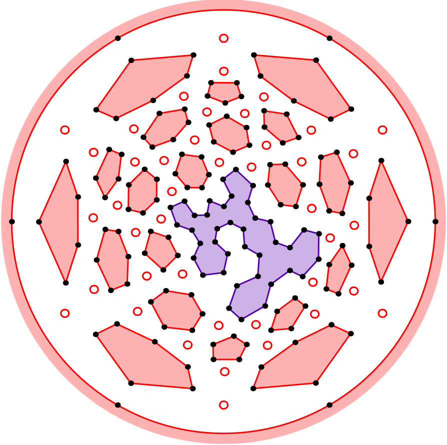

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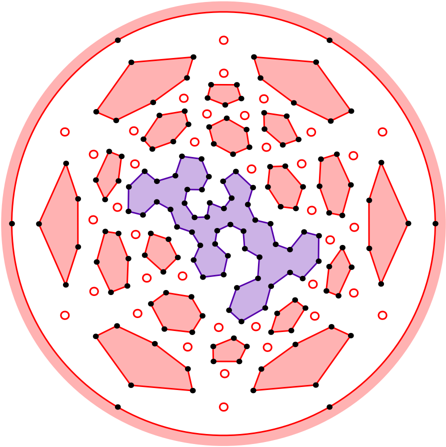

The construction revisited

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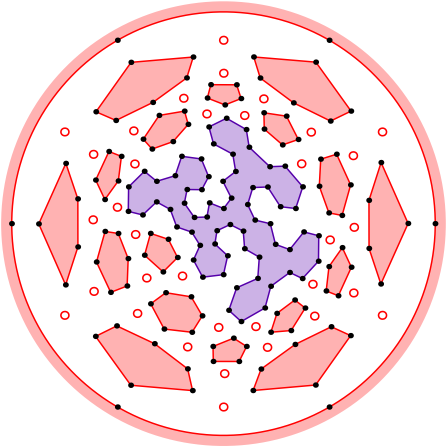

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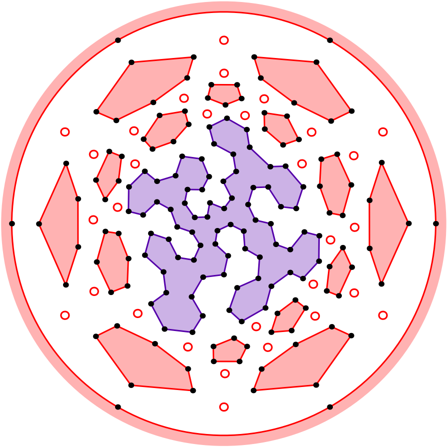

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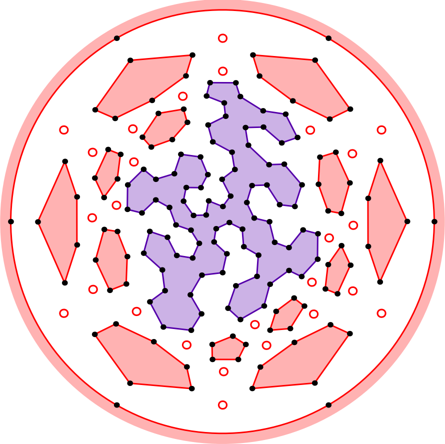

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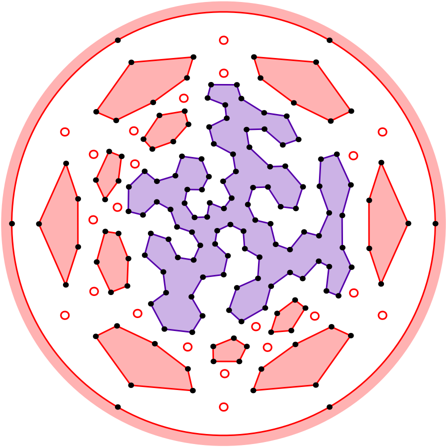

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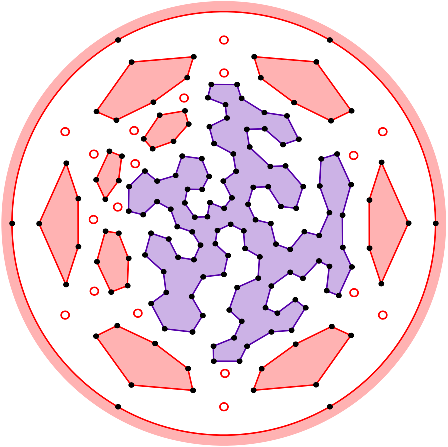

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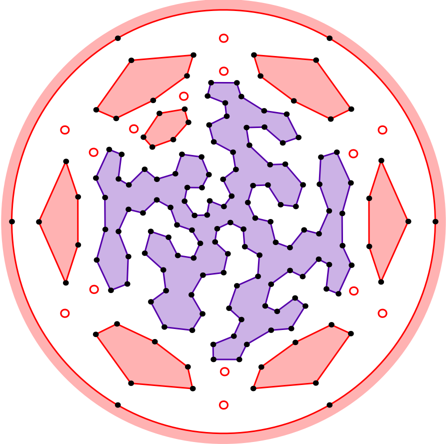

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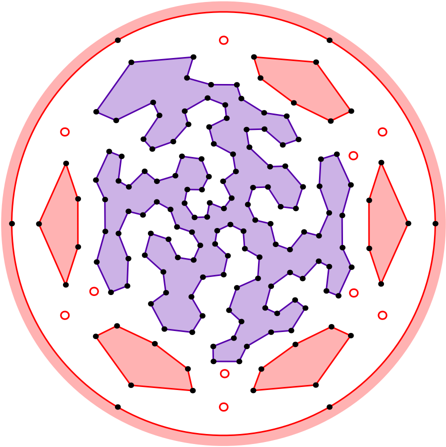

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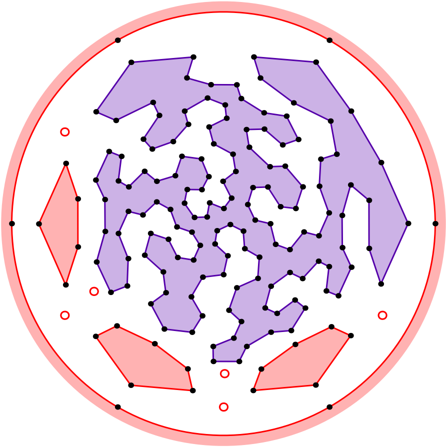

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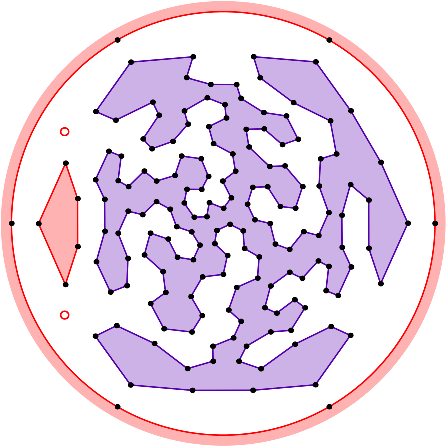

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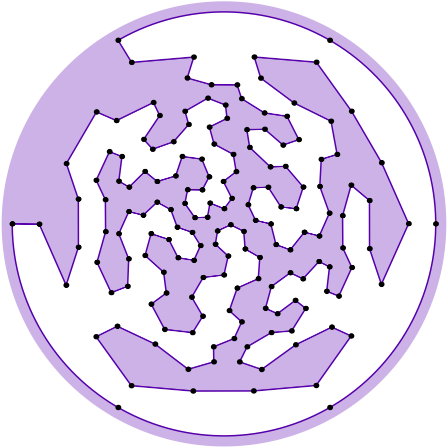

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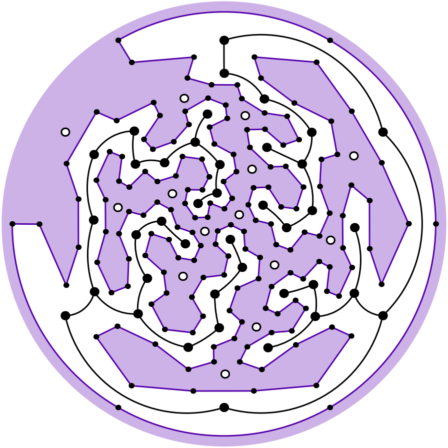

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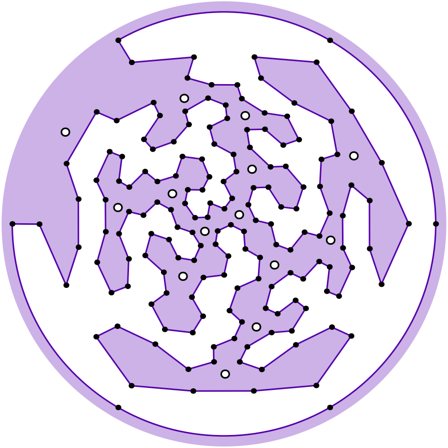

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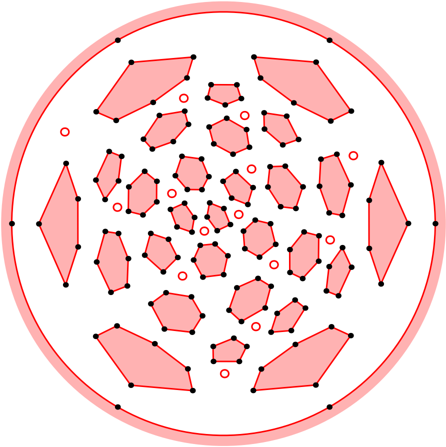

Can we do the same thing for any Barnette graph?

We introduce a linear-time algorithm that finds a Hamilton cycle in a Barnette graph:

- find a 2-factor such that no cycle is inside another
- use resonant hexagons to glue the cycles together

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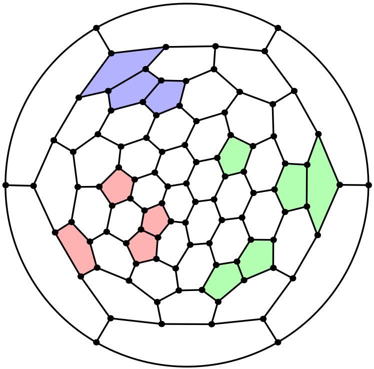
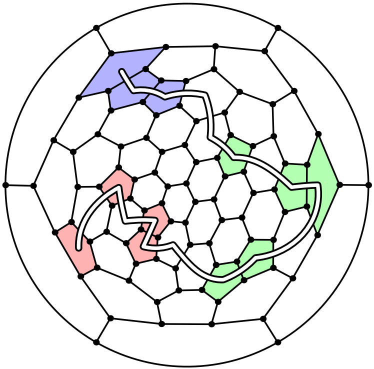
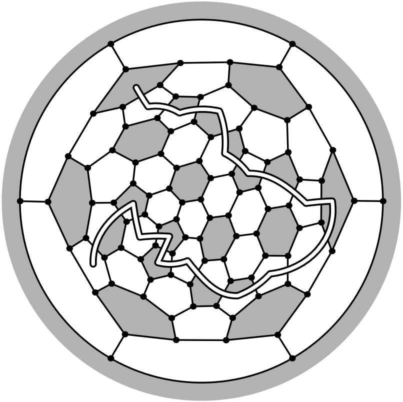
Identify the clusters of pentagons.



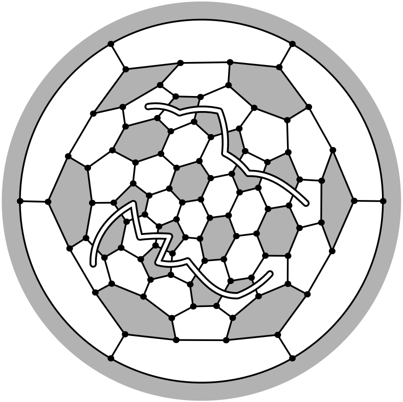
Find a path that meets all the pentagons and cut the graph.



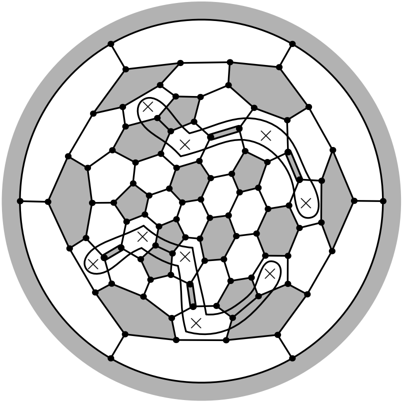
Choose one of the three colors.



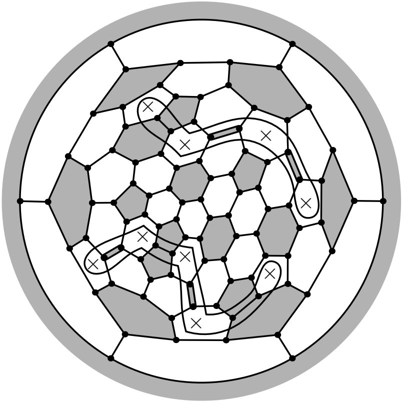
Wherever the two colorings are not compatible,



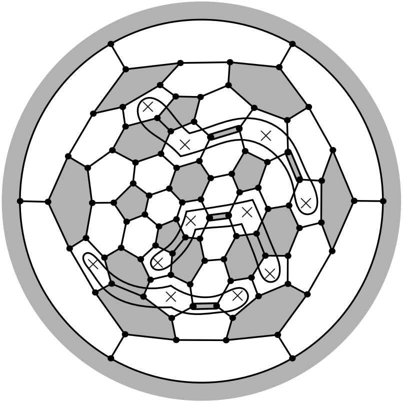
introduce non-resonant white faces.



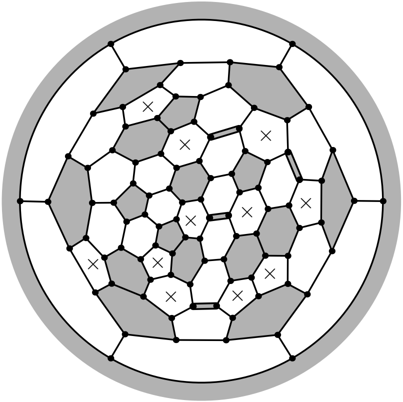
To change the parity of the number of cycles locally,



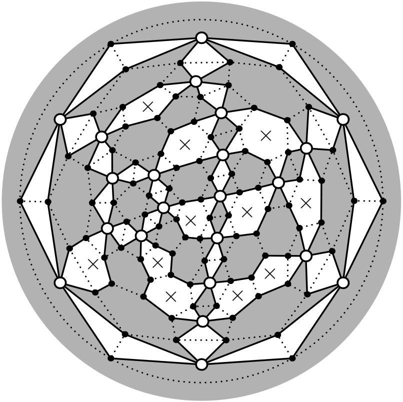
use a pair of pentagons.



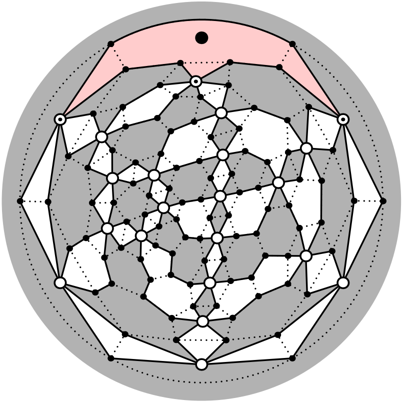
Introduce a new vertex for each resonant hexagon.



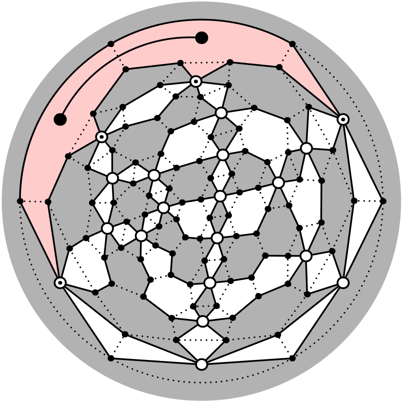
Introduce a new vertex for each resonant hexagon.



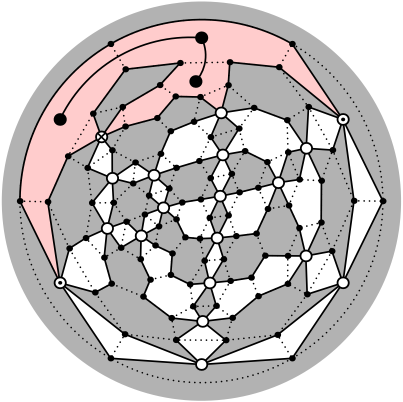
Choose a resonant hexagon to start with and color it red.



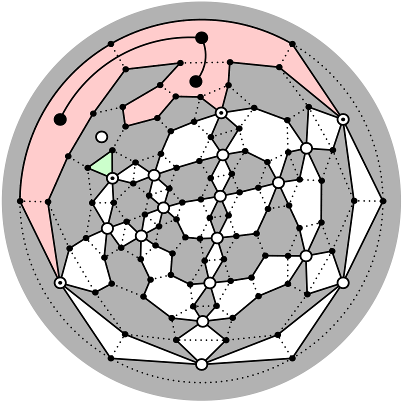
Propagate the red color if possible,



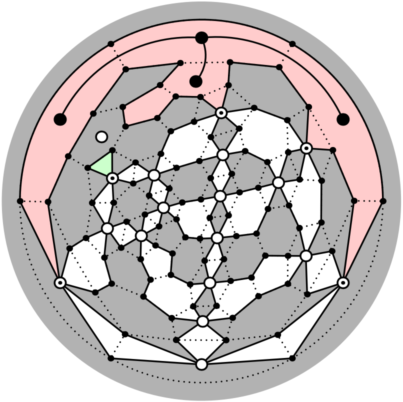
if not, glue three black cycles together.



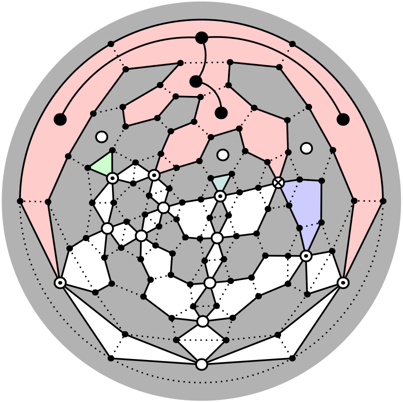
Continue...



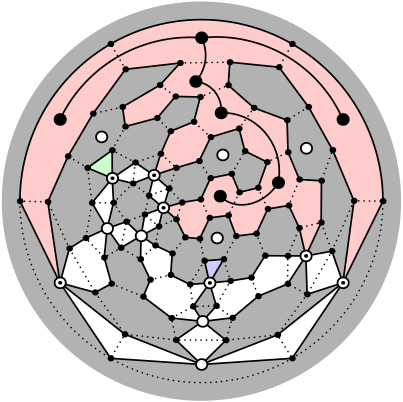
Continue...



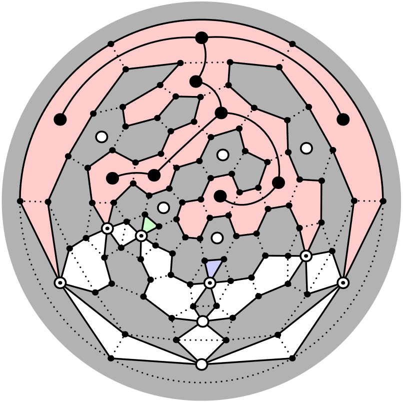
Continue...



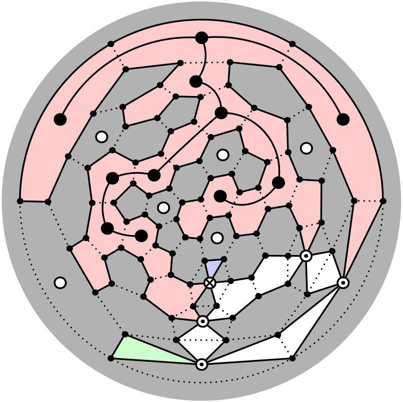
Continue...



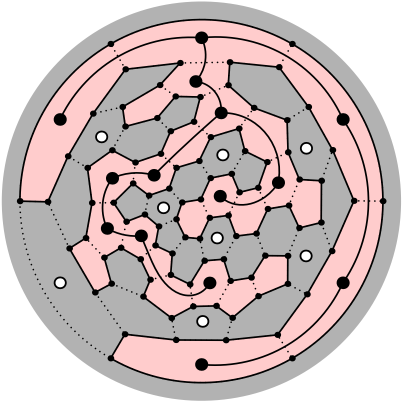
Continue...



Continue...



...until there is a single red and a single black face.



To prove the Barnette's conjecture, it suffices now to examine all the possible configurations of small faces (quadrangles and pentagons) at distance at most two and verify that the algorithm works locally (computer assisted part).

The numbers of clusters containing f_4 quadrangles and f_5 pentagons to be processed:

$f_4 \setminus f_5$	1	2	3	4	5	6	7	8	9
0	1	3	11	67	408	3015	18360	24521	980
1	3	22	167	1408	6452	5229	609		
2	16	183	401	218	54				

Thank you for your attention!