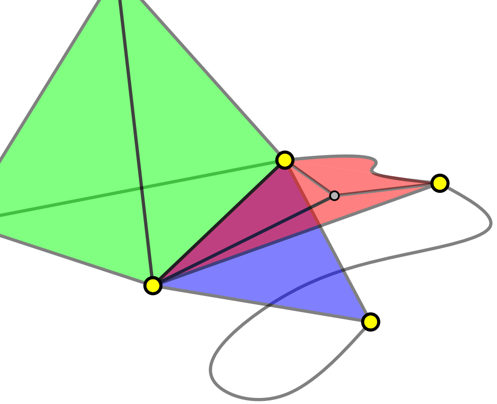
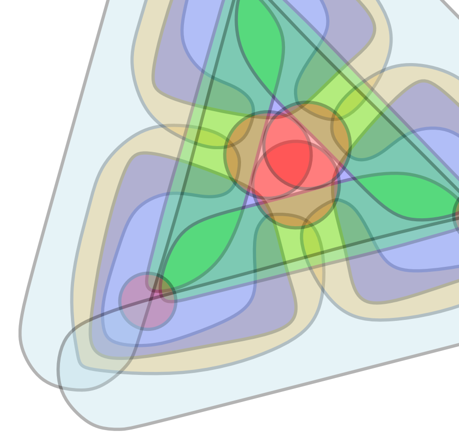
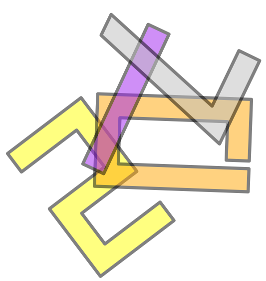
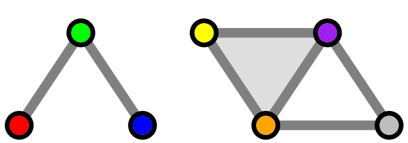




Helly-type theorem and topological combinatorics

Xavier Goaoc





”topological combinatorics”...

... what do **combinatorics** and **topology** have to do with one another?

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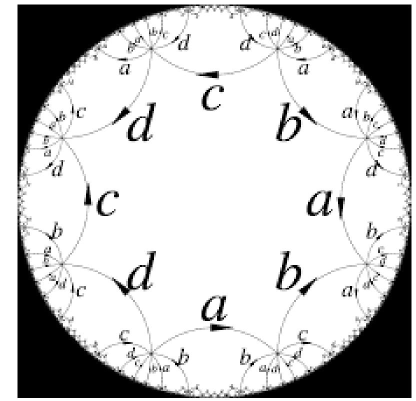
Combinatorics routinely used in topology.

Discrete approaches (eg. Whitehead's collapses)



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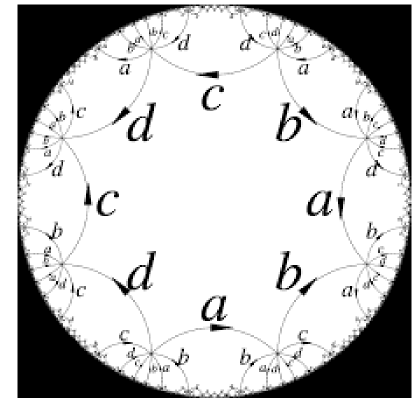

The use of topology in combinatorics is perhaps more surprising.

Győry and Lovász (1977-1978): any k-connected graph $G = (V, E)$ can be partitioned into k subsets that induce connected subgraph, have prescribed size (summing to $|V|$) and each contain a prescribed element.

Lovász (1978): if the n-elements subsets of the $\{1, 2, \ldots, 2n + k\}$ are partitioned into $k + 1$ classes then some class contains disjoint subsets.

Principle of a proof of the Kneser conjecture.

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Chromatic number of the graph $G = (V, E)$ where $V = \binom{[2n+k]}{n}$ and E connects disjoint subsets.

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Blueprint:

Associate to G some topological space X with an antipodality action via some "neighborhood" simplicial complex.

Translate c-colorings of G into antipodal maps $X \to \mathbb{S}^{c-2}$.

Bound the connectivity of X and apply the Borsuk-Ulam theorem.

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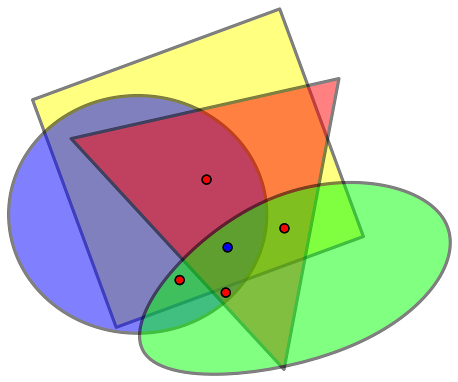
Examine similar approaches, for geometric questions, in a few more words...

Helly's Theorem. Any finite family of convex sets in $\mathbb{R}^d$ has non-empty intersection if every $d+1$ elements have non-empty intersection.



[Helly, 1913]

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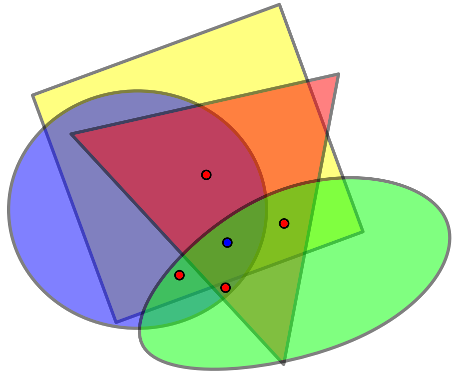

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This talk:

Examine Helly's theorem via simplicial complexes.

Relax "convexity" into more general topological conditions.

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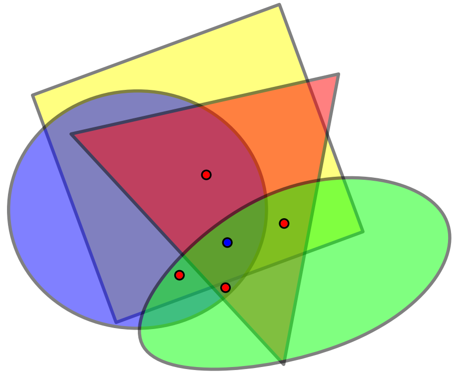

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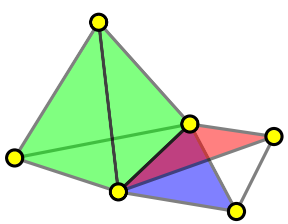
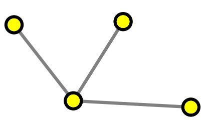
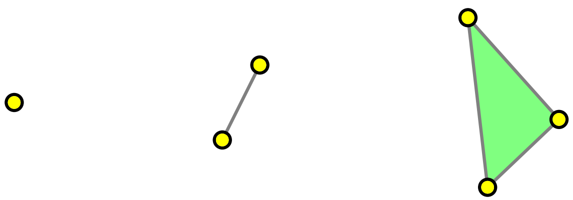
geometric simplicial complex

"A collection of geometric simplices in $\mathbb{R}^d$ closed by taking faces and such that any two are disjoint or intersect in a common face."





A geometric simplex is the convex hull of $k \leq d+1$ affinely independent points (= the vertices) in $\mathbb{R}^d$. A face of a simplex is the convex hull of a subset of its vertices.



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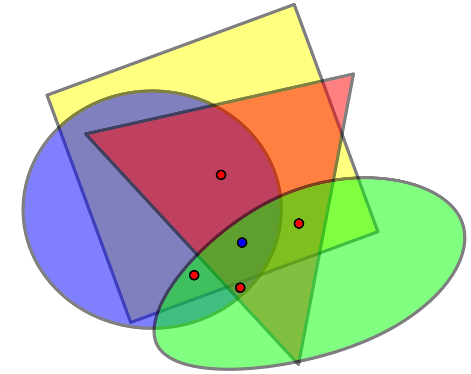

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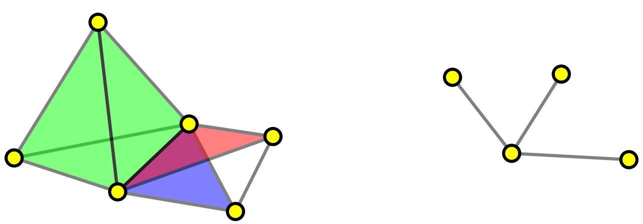

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"A collection of sets that is closed under taking subsets."

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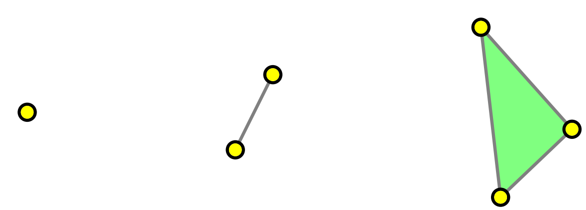

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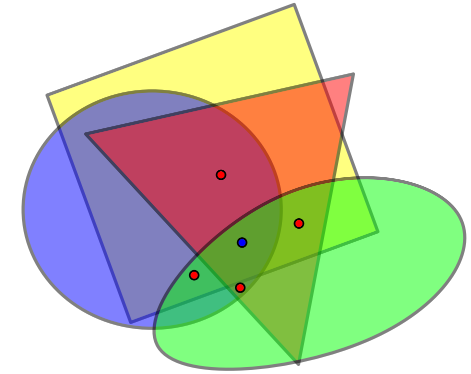

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Examine Helly's theorem via simplicial complexes.
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collect subsets of vertices forming a simplex

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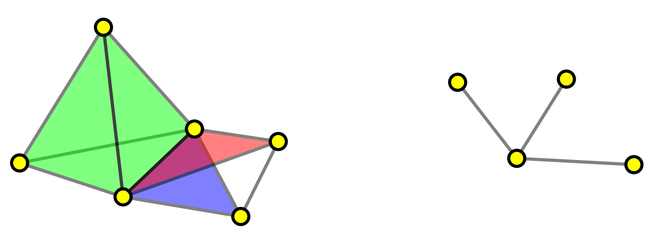

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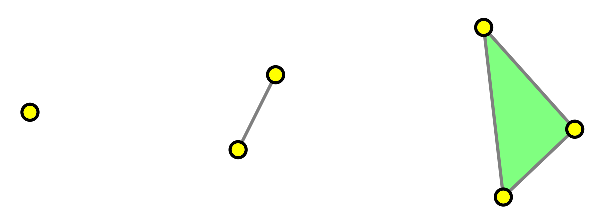

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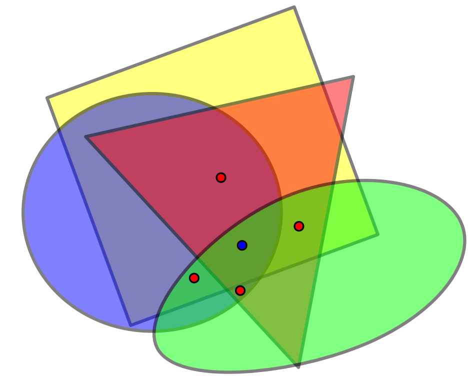

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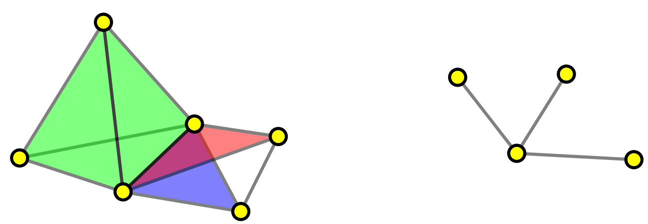

$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}\}$

map singletons to points in general position in $\mathbb{R}^d$, d large enough

take convex hulls

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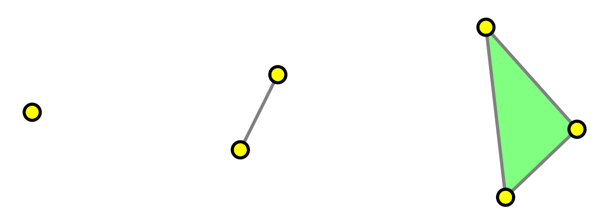
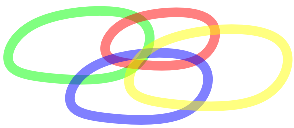


Why topological generalizations of Helly's theorem?

⋆ "Sort out" an industry of Helly-type theorems...



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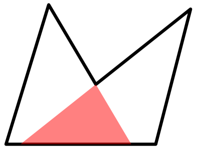

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Common eigen vector to finitely many linear operators in $\mathbb{K}^d$

$\lfloor \frac{3d}{2} \rfloor$ [Polyanskii 2016]

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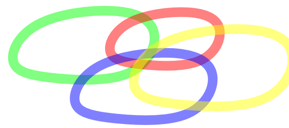

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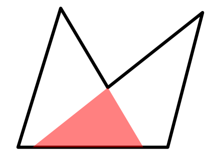

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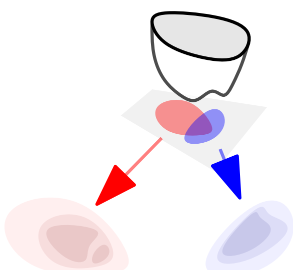
Study the combinatorial aspects of optimization problems over continuous sets.



Given $f : \mathbb{R}^d \to \mathbb{R}$ and subsets $C_1, C_2, \ldots, C_n \subset \mathbb{R}^d$, minimize f over $\bigcap_{i=1}^{n} C_i$.

Define $F_t = \{C_1, C_2, \ldots, C_n\} \cup \{f^{-1}((-\infty, t))\}$.

$\inf_{\cap_{i=1}^n C_i} f = \inf \{t \in \mathbb{R} : \cap_{A \in F_t} A \neq \emptyset\}$.

Uniform Helly-type theorem for the $F_t \Rightarrow O(n)$-time algorithm to minimize f.

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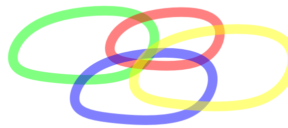

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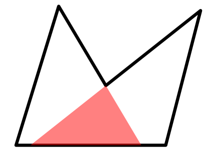

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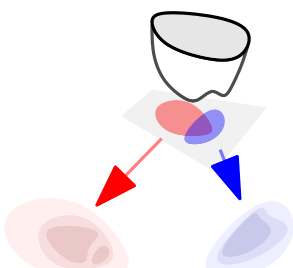

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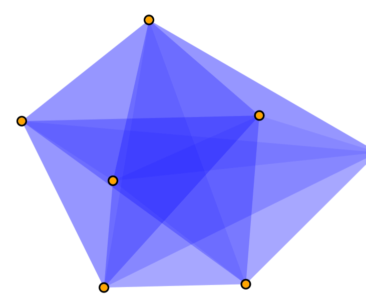
⋆ Foundation of combinatorial convexity, for instance...

Fractional Helly Theorem. For any $0 < \alpha \leq 1$ and any finite family of n convex sets in $\mathbb{R}^d$, if $\alpha\binom{n}{d+1}$ of the $(d+1)$-elements subsets of the family have non-empty intersection then some $(1-(1-\alpha)^{\frac{1}{d+1}})n$ elements of the family have non-empty intersection.

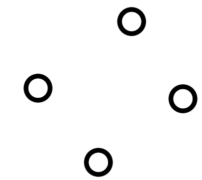
First selection lemma. For any finite point set $P \subset \mathbb{R}^2$ there exists a point $x \in \mathbb{R}^2$ contained in at least $\frac{2}{9}\binom{|P|}{3}$ triangles spanned by P.

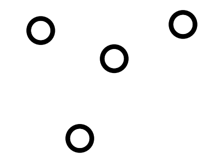


Holds in any dimension with $\frac{2}{9}$ replaced by suitable constant.

1. Non-embeddable complexes

Radon's lemma. Any $n \geq d+2$ points in $\mathbb{R}^d$ can be partitioned into two subsets whose convex hulls intersect.





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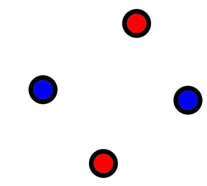

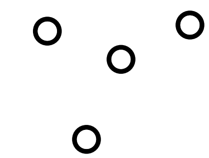

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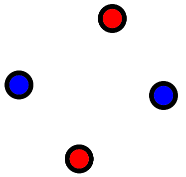

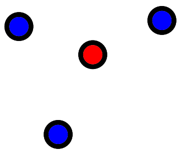

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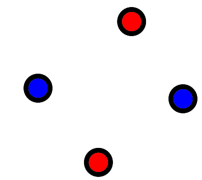
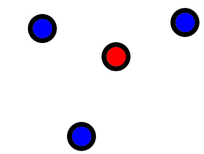


Proof: Fix a frame, consider $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$ and define $\mathbf{x}_i := \begin{pmatrix} x_i \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$.

Consider a non-trivial linear dependency $\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = \vec{\mathbf{0}}$.

Split it: $\sum_{i:\alpha_i \geq 0} \alpha_i \mathbf{x}_i = \sum_{i:\alpha_i < 0} -\alpha_i \mathbf{x}_i$.



Last coordinate ensures that $\sum_{i:\alpha_i \geq 0} \alpha_i = \sum_{i:\alpha_i < 0} -\alpha_i$. $\square$

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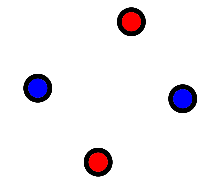

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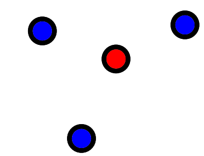

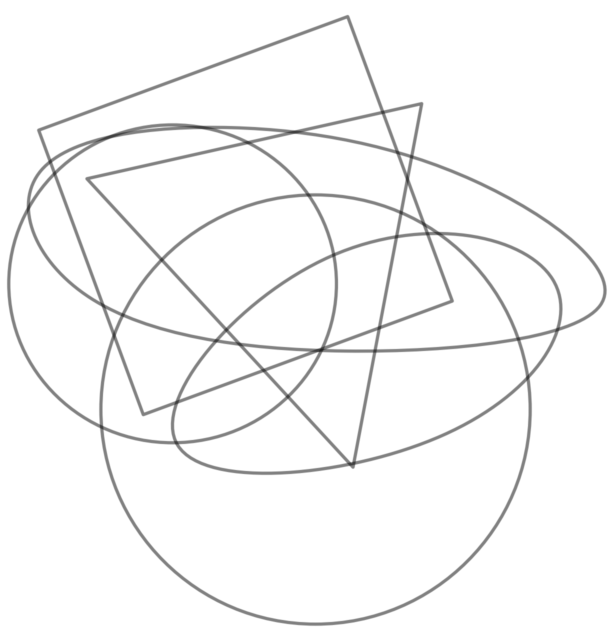

Classical proof of Helly's theorem from Radon's lemma.

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^2$ where any three intersect.

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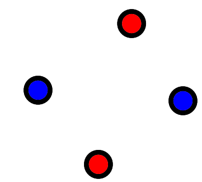

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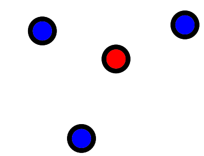
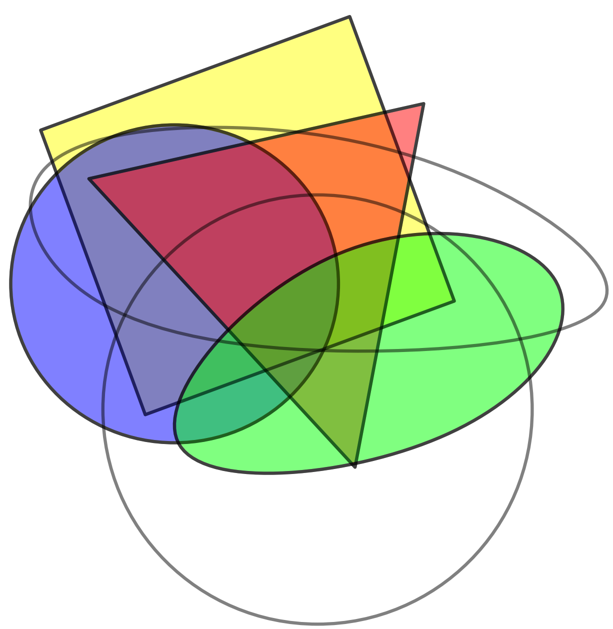


Last coordinate ensures that $\sum_{i:\alpha_i \geq 0} \alpha_i = \sum_{i:\alpha_i < 0} -\alpha_i$. $\square$

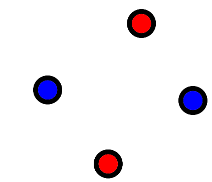


Classical proof of Helly's theorem from Radon's lemma.

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^2$ where any three intersect.

Consider four of them $A_1, \ldots, A_4$

Radon's lemma. Any $n \geq d+2$ points in $\mathbb{R}^d$ can be partitioned into two subsets whose convex hulls intersect.



Proof: Fix a frame, consider $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$ and define $\mathbf{x}_i := \begin{pmatrix} x_i \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$.

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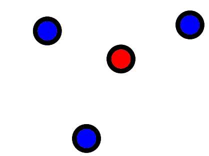

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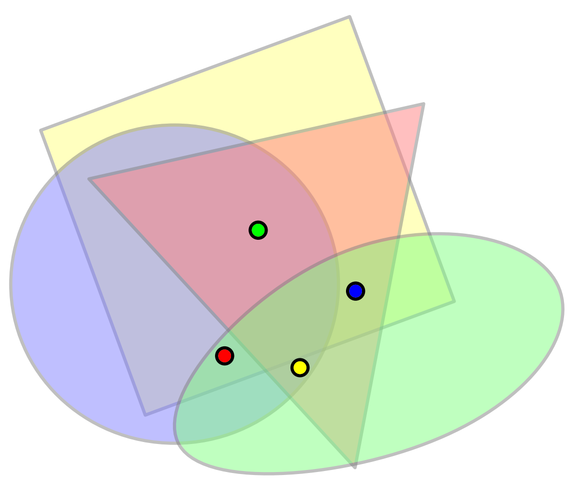

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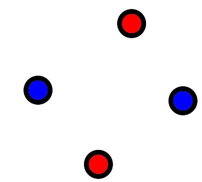

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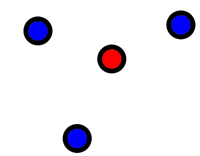

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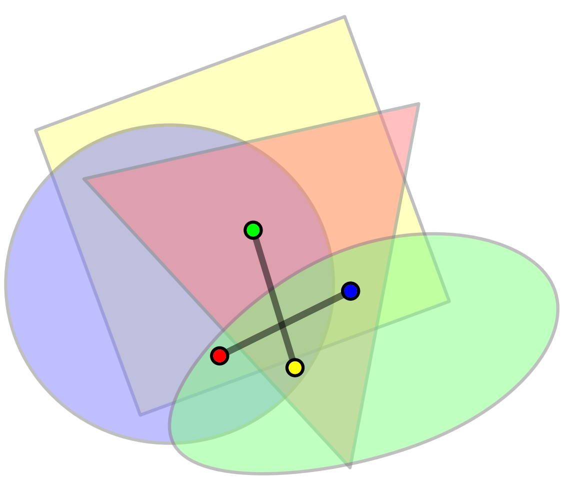

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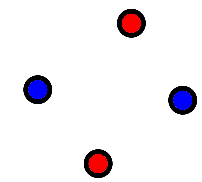

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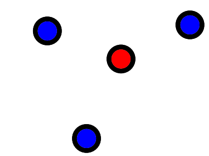

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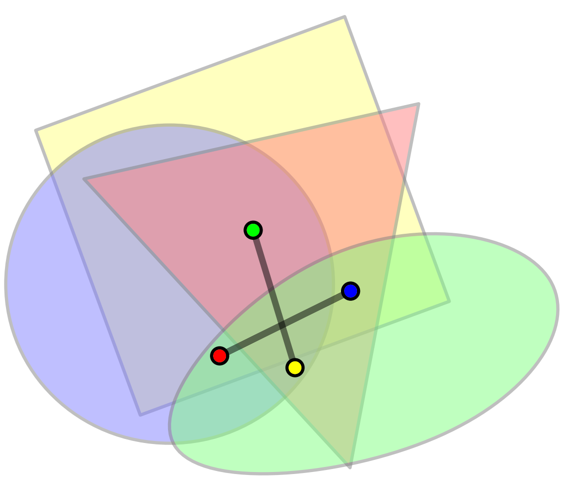

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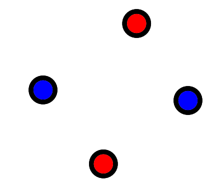

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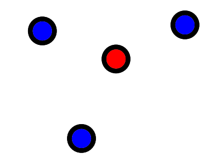

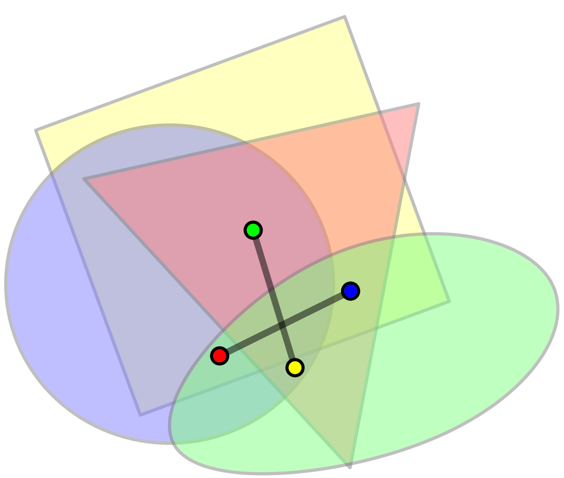

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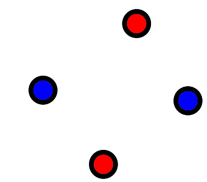

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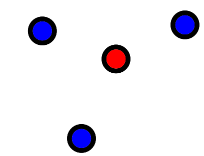

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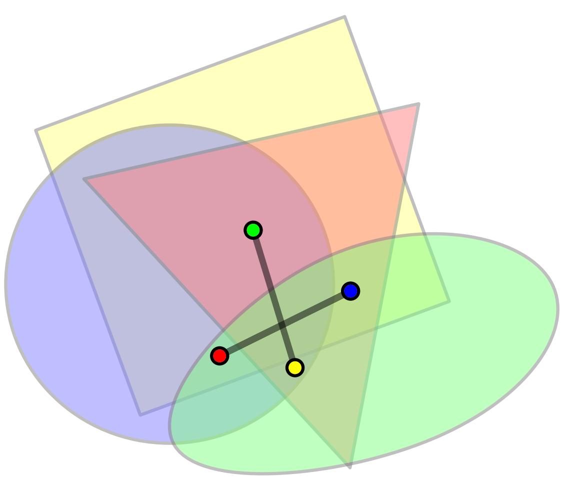

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So any four intersect. Consider five of them...

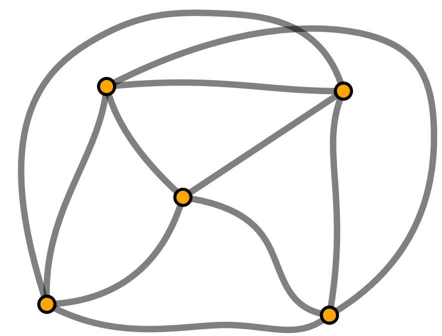
This is a non-embeddability argument.

Same argument, replacing Radon's lemma by the non-planarity of K_5.

Fact. In any (topological) drawing of K_5 in the plane there are two vertex-disjoint edges whose images intersect.

K_5 is the complete graph on 5 vertices.

Vertices are mapped to points,
edges to Jordan arcs joining the image of their vertices.



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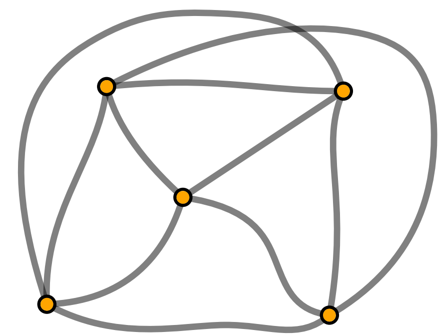

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Corollary. Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^2$ such that every subfamily has empty or path-connected intersection. If every four intersect then they all intersect.

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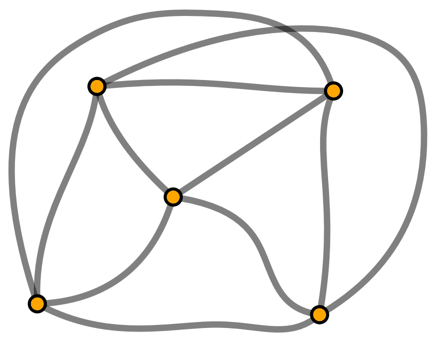

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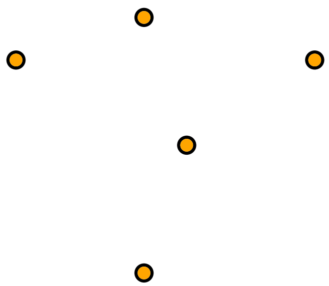

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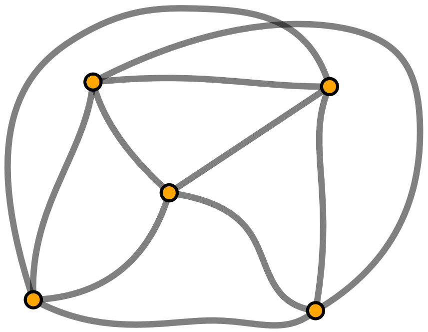

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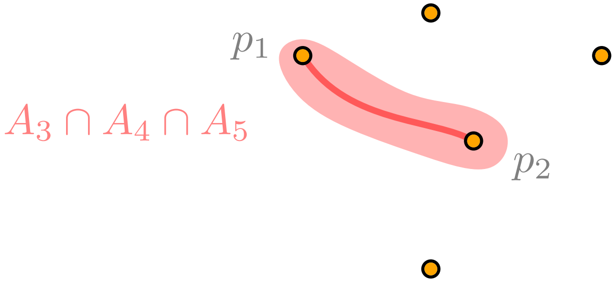

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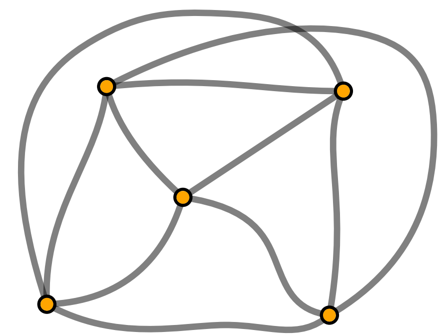

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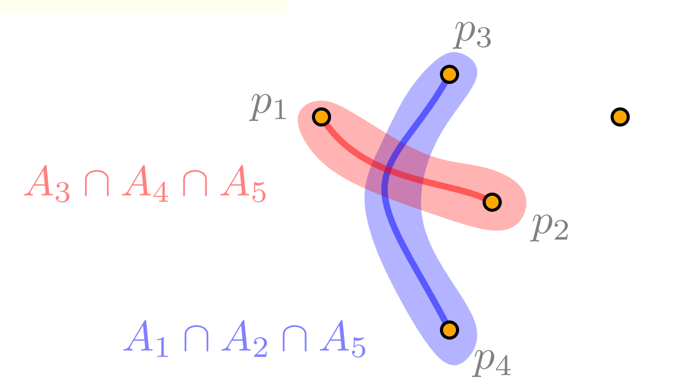

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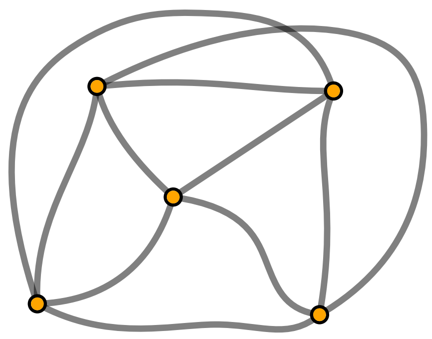

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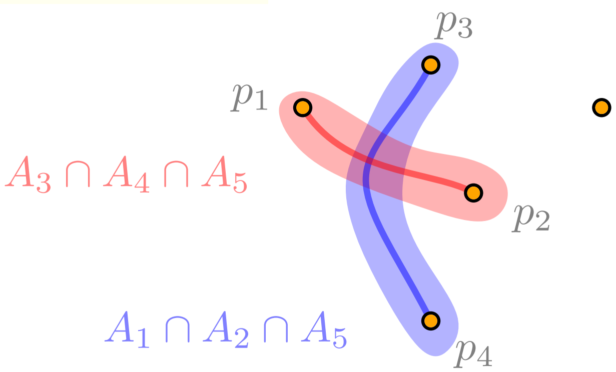
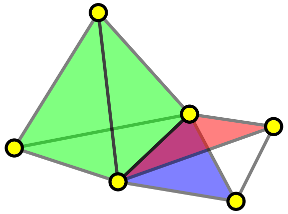


This idea generalizes to **higher dimension**.



Any abstract simplicial complex K has a geometric realization $|K|$.

Map singletons $x \in K$ to points $p(x)$ in general position in $\mathbb{R}^d$, d large enough.

For every $\sigma \in K$ let $|\sigma| \stackrel{def}{=} \mathrm{conv}\{p(x) : x \in \sigma\}$, and $|K| \stackrel{def}{=} \cup_{\sigma \in K} |\sigma|$.

topological space defined up to homeomorphism (choices of d and $p(x)$).

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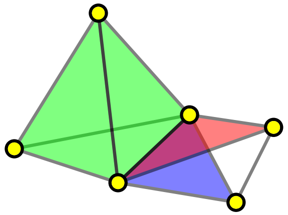

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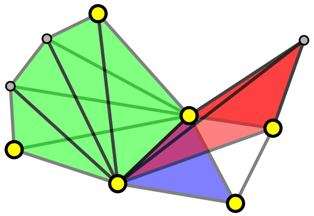
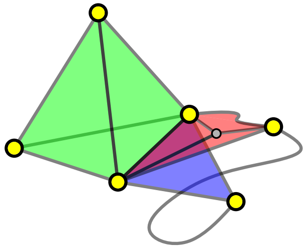


Generalization of planarity: existence of an **injective map** $|K| \to \mathbb{R}^m$.



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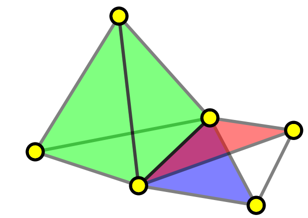

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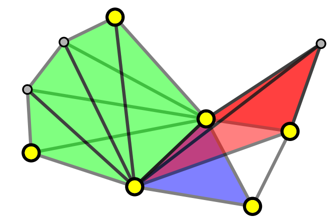

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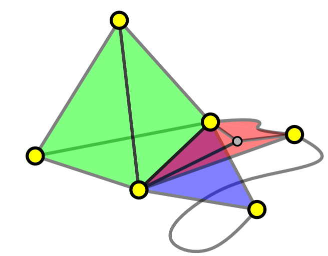

Theorem. Let $\Delta_n^{(k)}$ be the k-dimensional skeleton of the n-dimensional simplex. For any continuous map $f : |\Delta_{2k+2}^{(k)}| \to \mathbb{R}^{2k}$ there exist $\sigma, \tau \in \Delta_{2k+2}^{(k)}$ such that $\sigma \cap \tau = \emptyset$ and $f(\sigma) \cap f(\tau) \neq \emptyset$.

[Van Kampen, 1932][Flores, 1933]

$\Delta_n^{(k)} \overset{\text{def}}{=} \binom{[n+1]}{\leq k+1}$ = *all faces of dimension up to k over $n+1$ vertices.*

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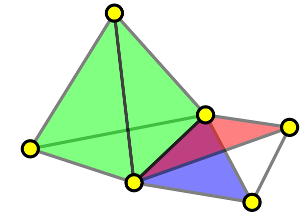

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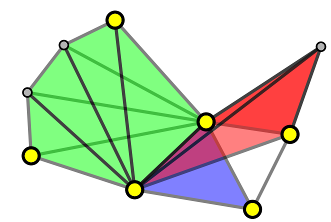

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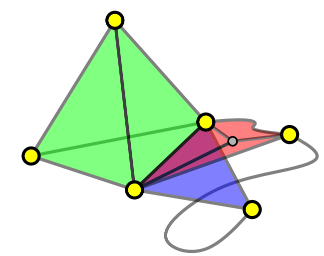

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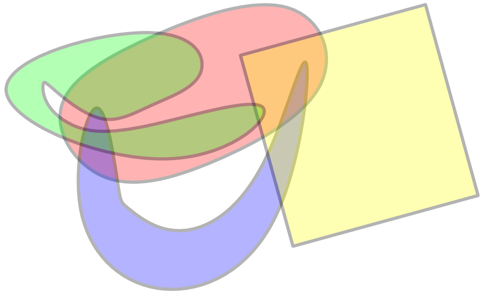
Corollary. Let $\mathcal{F}$ be a family of at least $2k+3$ subsets of $\mathbb{R}^{2k}$ or $\mathbb{R}^{2k-1}$ such that every subfamily has $(k-1)$-connected intersection. If every $k+1$ intersect then they all intersect.

This argument can be pushed a bit further (with help from Ramsey).



Now consider a family of subsets of $\mathbb{R}^2$ where every subfamily intersects in at most two path-connected components.

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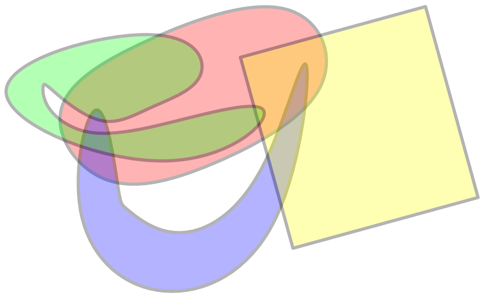

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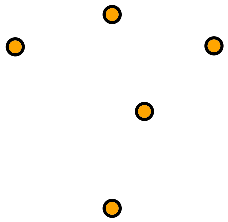

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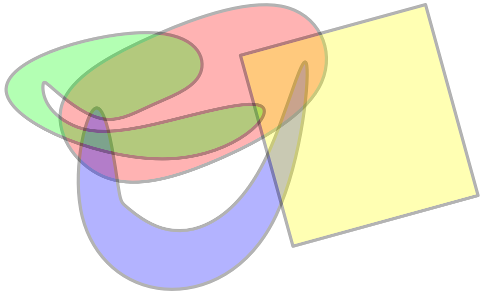

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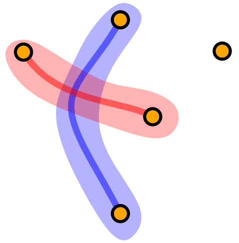

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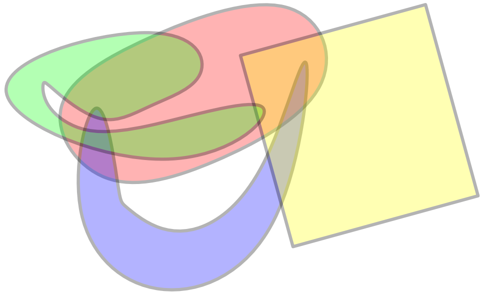

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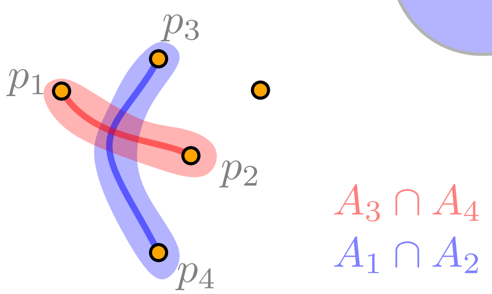

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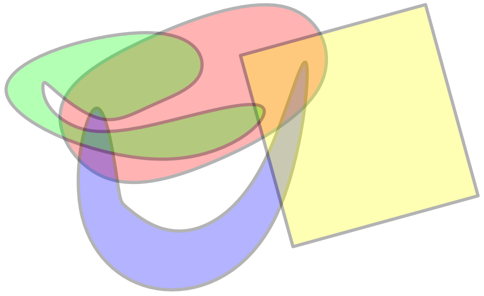

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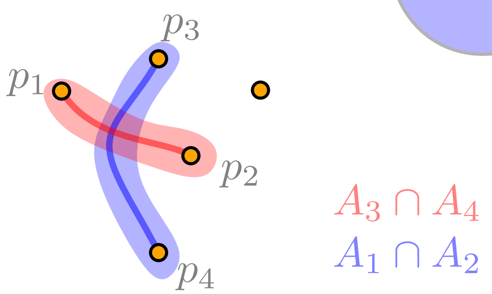

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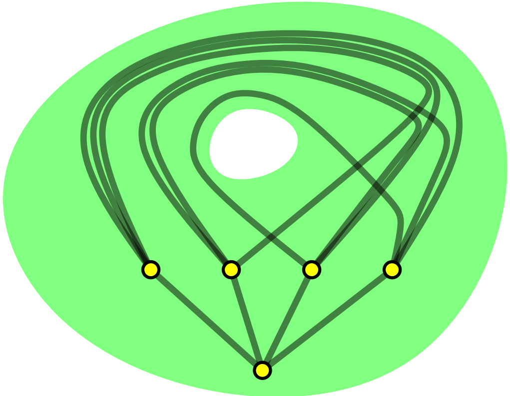

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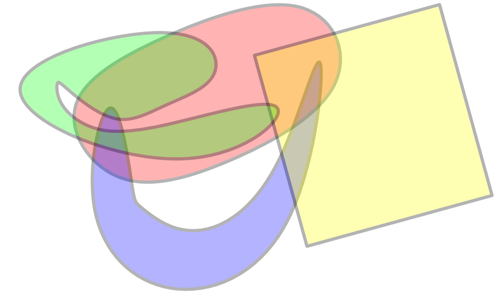

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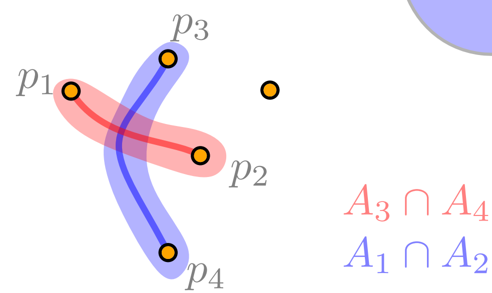

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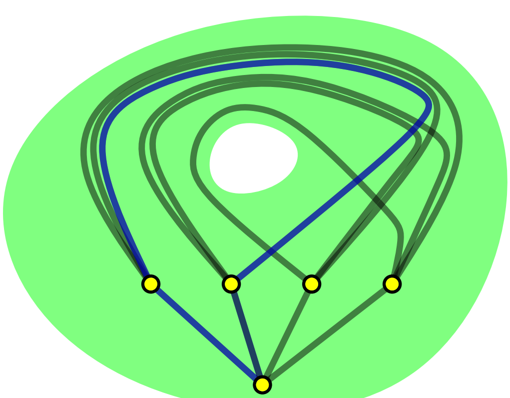

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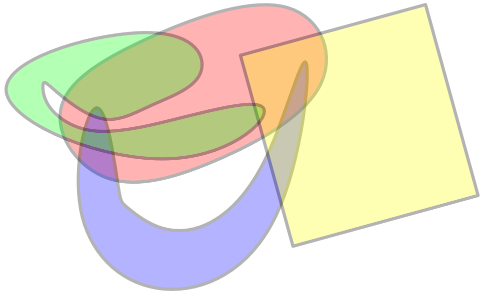

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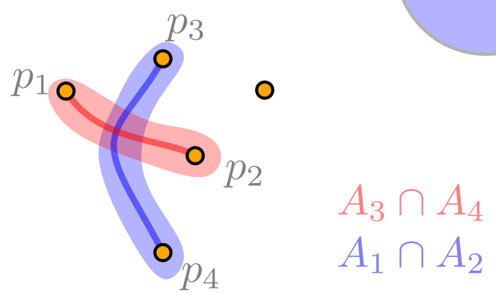

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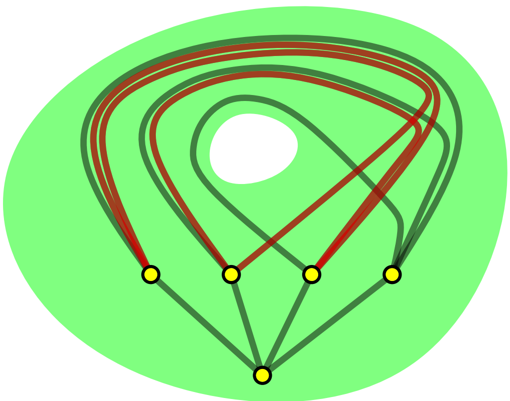

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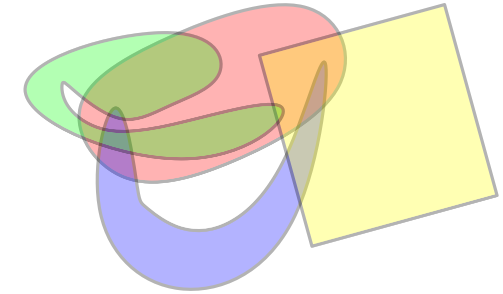

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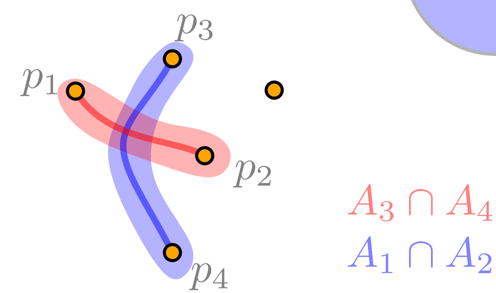

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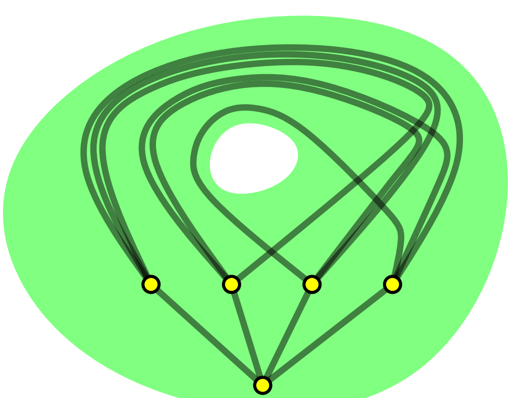

And quite a bit further (using homological relaxations of embeddings + Ramsey on homology classes).

Theorem. The Helly number of any family of subsets of $\mathbb{R}^d$ can be bounded by a function of d and

$$\max_{G \subseteq \mathcal{F}} \max_{i \leq \lceil \frac{d}{2} \rceil - 1} \widetilde{\beta}_i(\cap G, \mathbb{Z}_2).$$

[Matoušek, 1996][G.-Patak-Patáková-Tancer-Wagner, 2015]

2. Nerve complexes

The nerve $\mathcal{N}(\mathcal{F})$ of a family $\mathcal{F}$ of sets is $\mathcal{N}(\mathcal{F}) = \{\mathcal{G} : \mathcal{G} \subseteq \mathcal{F}$ and $\cap\, \mathcal{G} \neq \emptyset\}$.

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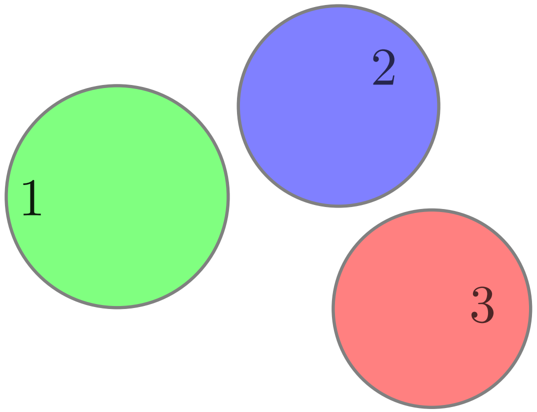

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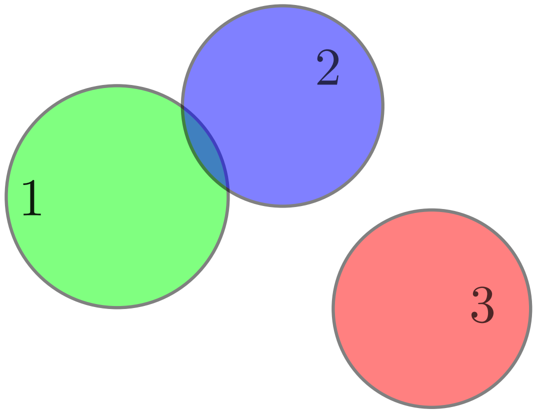

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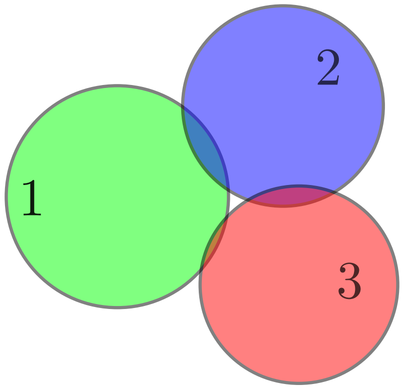

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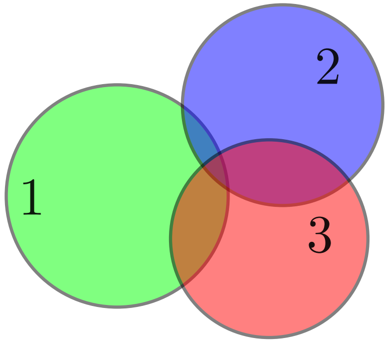

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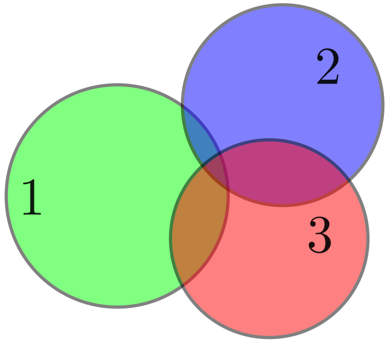

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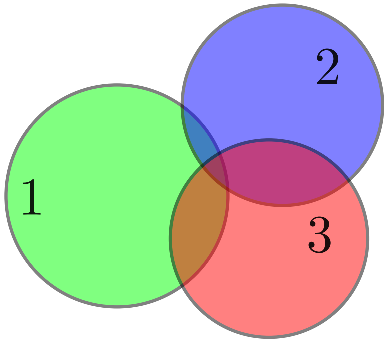

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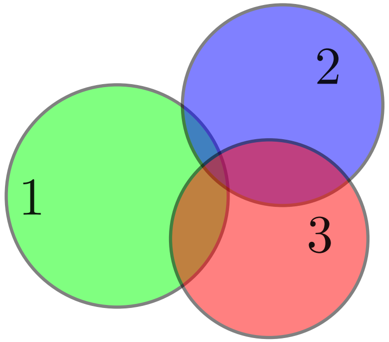
$\mathcal{F}$ 

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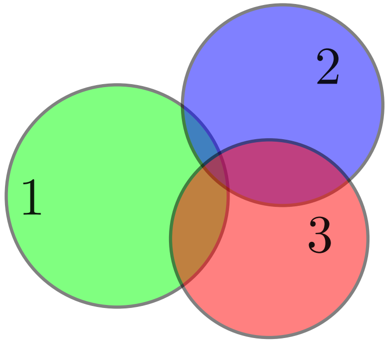
$\mathcal{F}$ 

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so...

Families eluding Helly-type theorems have nerves with "simplicial holes" of arbitrary high dimensions.

Statement about collections of families, eg. "all finite families of convex sets in $\mathbb{R}^2$".
Dimension of simplicial holes VS size of family.

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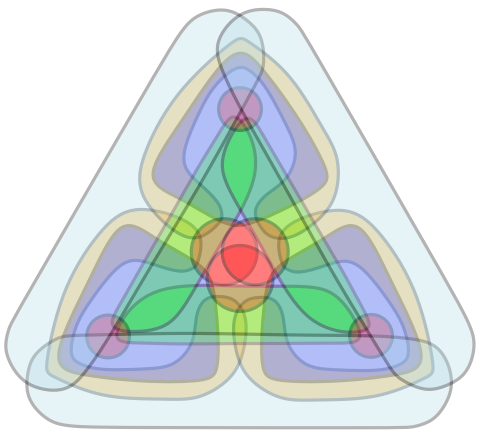

Families of sets where subfamilies have "hole-free" intersections.

Good cover: subfamilies have empty or contractible intersections.

Contractible = homotopic to a point.

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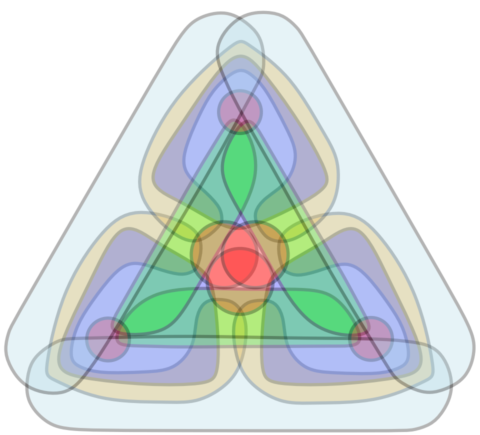

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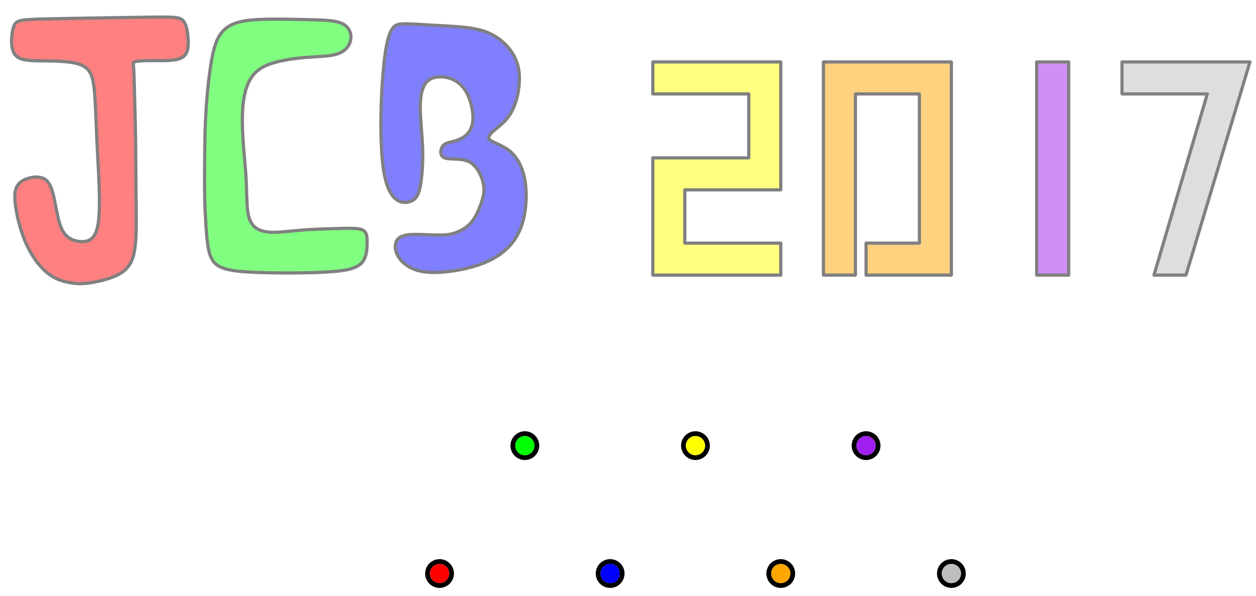

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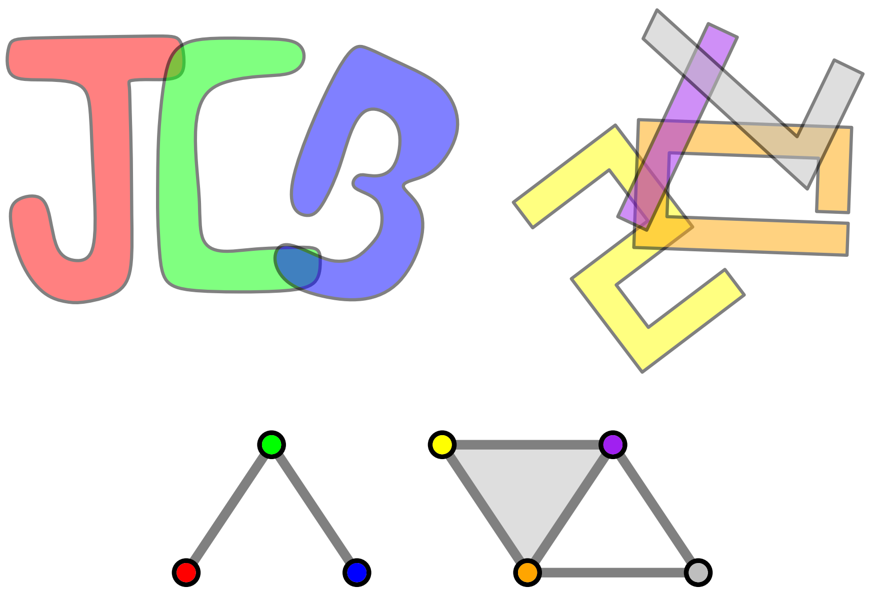

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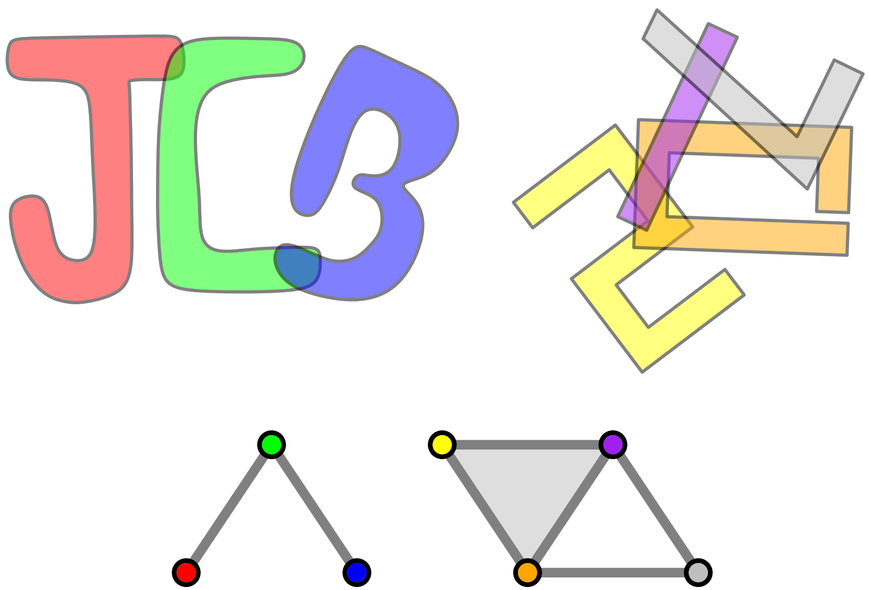

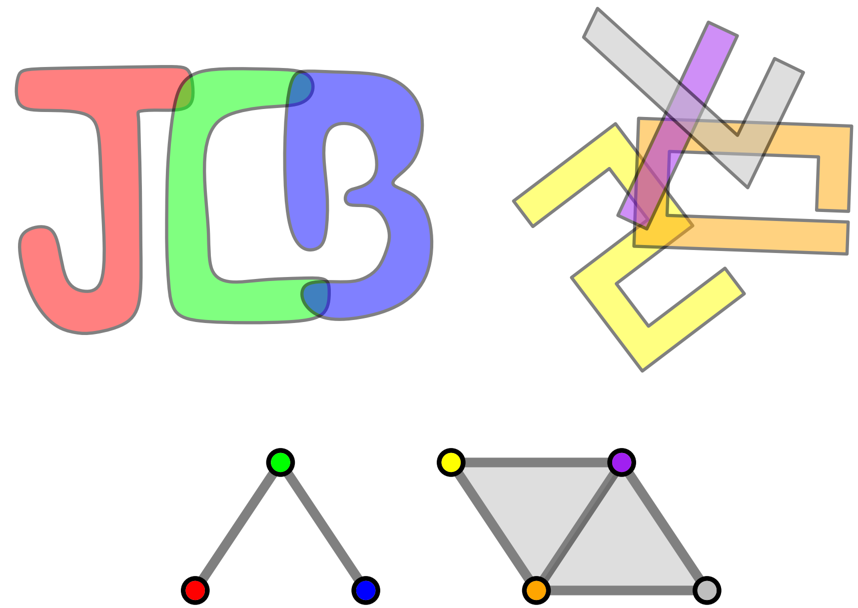

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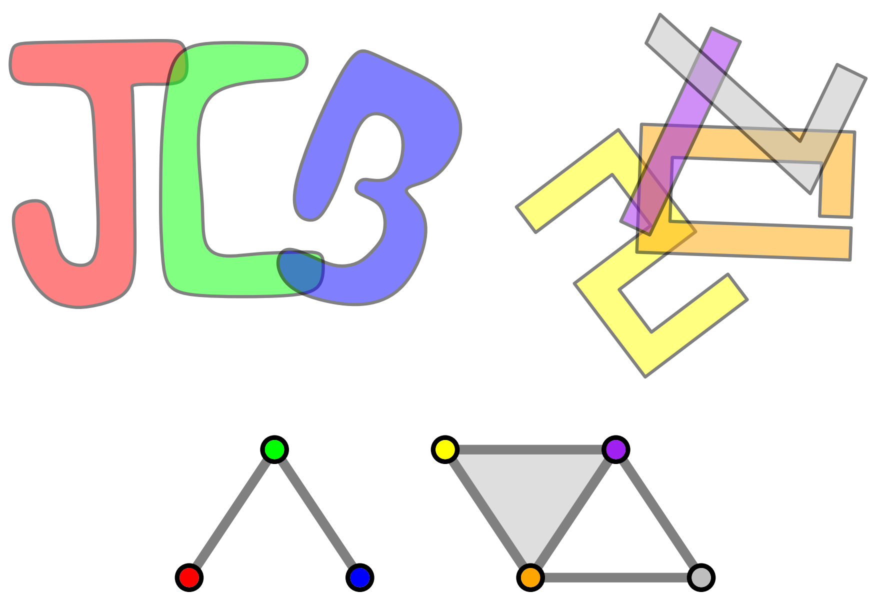
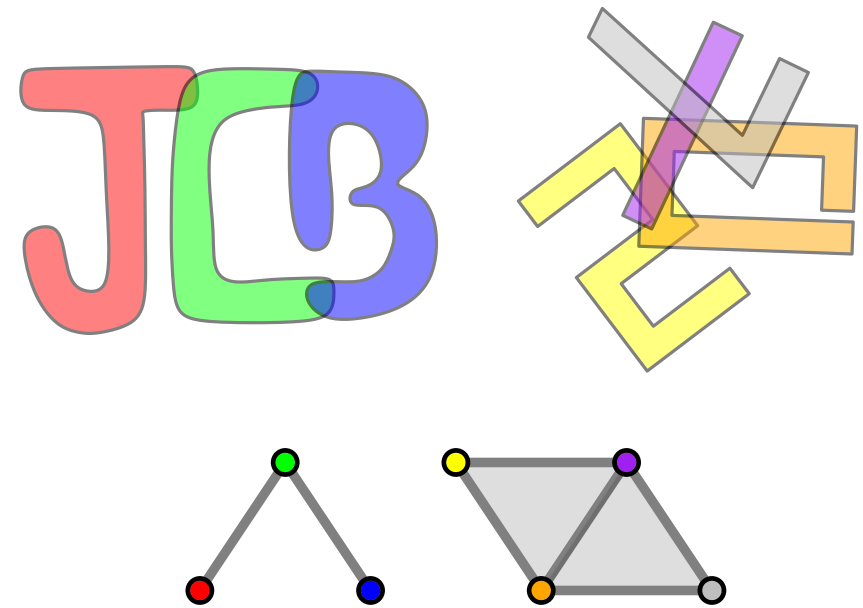
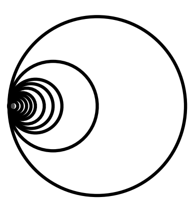






Fact. Any open subset $\mathbb{R}^d$ has trivial $\mathbb{Q}$-homology in dimension $> d$.

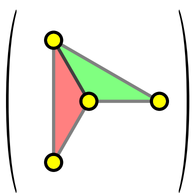
Homological Helly Theorem. Let $\mathcal{F}$ be an acyclic cover in $\mathbb{R}^d$. If every $d+1$ elements of $\mathcal{F}$ have nonempty intersection then $\mathcal{F}$ has nonempty intersection.

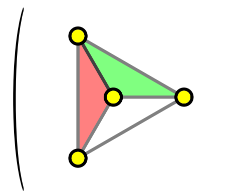
The Leray number of a simplicial complex K with vertex set V is

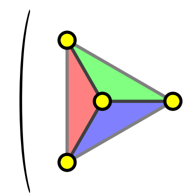
$$L(K) = \min\{\ell \in \mathbb{N} : \quad \forall i \geq \ell, \forall S \subseteq V, \quad \tilde{H}_i(K[S], \mathbb{Q}) = 0\}$$

$K[S] = \{\sigma : \sigma \in K$ *and* $\sigma \subseteq S\}$ *is the subcomplex induced on* K *by* S

Maximum dimension of non-trivial homology in an induced subcomplex

L 

L 

L 

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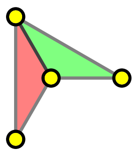
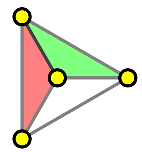
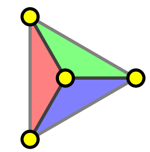
L () $= 1$

L ()

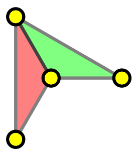
L ()

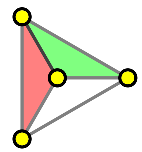
The Leray number of a simplicial complex K with vertex set V is

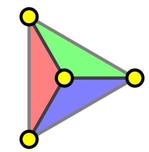
$$L(K) = \min\{\ell \in \mathbb{N} : \quad \forall i \geq \ell, \forall S \subseteq V, \quad \tilde{H}_i(K[S], \mathbb{Q}) = 0\}$$

$K[S] = \{\sigma : \sigma \in K$ and $\sigma \subseteq S\}$ is the subcomplex induced on K by S

Maximum dimension of non-trivial homology in an induced subcomplex

L $\Bigg($  $\Bigg) = 1$

L $\Bigg($  $\Bigg) = 2$

L $\Bigg($  $\Bigg)$

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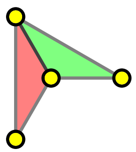
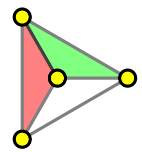
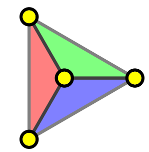
L () $= 1$

L () $= 2$

L () $= 2$

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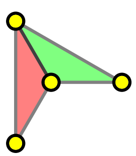
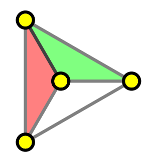
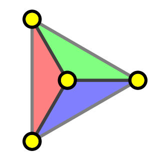
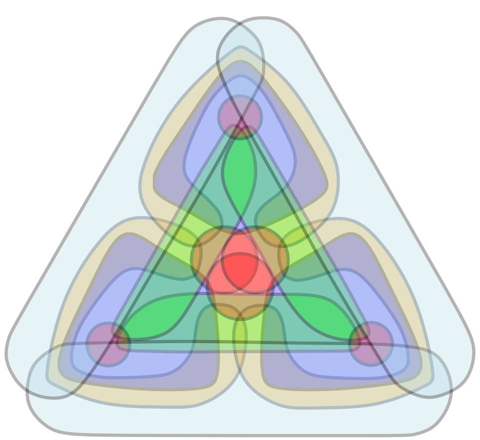
L () $= 1$ L () $= 2$ L () $= 2$

Nerve theorem $\Rightarrow$ Nerves of acyclic covers in $\mathbb{R}^d$ have Leray number $\leq d$.

Induced subcomplex of a nerve = nerve of a subfamily.



Theorem. If $L(K) \leq d$ then for any $r \geq 0$,

$$f_d(K) > \binom{n}{d+1} - \binom{n-r}{d+1} \Rightarrow f_{d+r}(K) > 0.$$

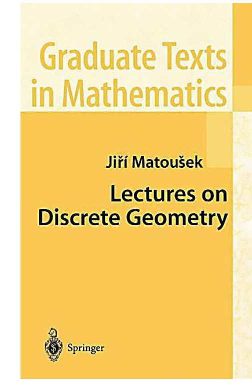
[Eckhoff 1984] [Kalai 1984]

$n = \#$ vertices and $f_i(K) = \#$ i-dimensional faces of K.

Many low-dimensional faces $\Rightarrow$ existence of high-dimensional faces.

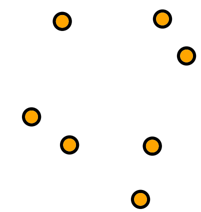
Sharp: r copies of $\mathbb{R}^d$ and $n-r$ hyperplanes in general position.

Excerpt from the theory of combinatorial convexity.



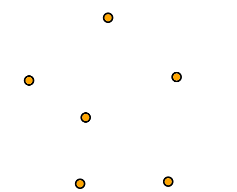
Fractional Helly theorem. For any $0 < \alpha \leq 1$ there exists $\beta > 0$ such that any finite family $\mathcal{F}$ of convex sets of $\mathbb{R}^d$ some $\alpha\binom{\mathcal{F}}{d+1}$ of the $(d+1)$-tuples intersect, then some $\beta|\mathcal{F}|$ of the sets intersect.

Tverberg's theorem. Any $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into r parts so that the convex hulls of these parts intersect.



First selection lemma. For any $d \geq 1$ There exists $c_d > 0$ such that for any set P of n points in $\mathbb{R}^d$, a fraction c_d of the d-simplices spanned by P intersect.

Weak ϵ-nets theorem. For every d and $\epsilon > 0$ there exists a constant $f(d,\epsilon)$ such that for any finite point set P in $\mathbb{R}^d$, there exists a subset $N \subseteq \mathbb{R}^d$ with $|N| \leq f(d,\epsilon)$ such that any convex set C with $|C \cap P| \geq \epsilon|P|$ must intersect N.



(p,q) theorem. For any family of convex sets in $\mathbb{R}^d$ where among any p, some q intersect, there exist $c(p,q,d)$ points that intersect every member of the family.

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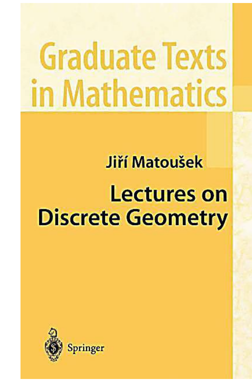

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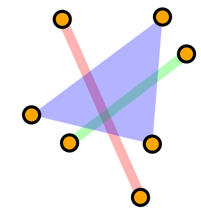

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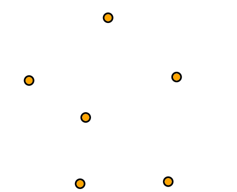

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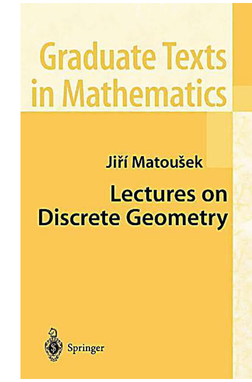

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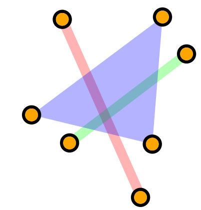

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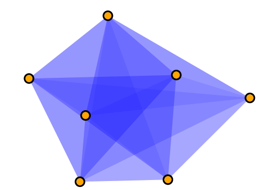

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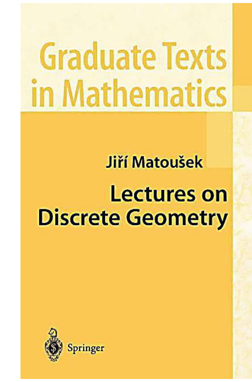

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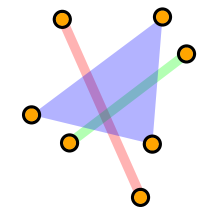

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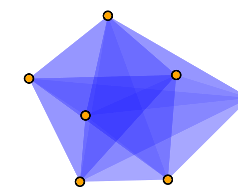

(p,q) **theorem.** For any family of convex sets in $\mathbb{R}^d$ where among any p, some q intersect, there exist $c(p,q,d)$ points that intersect every member of the family.

The Kalai-Eckhoff upper bound implies the fractional Helly theorem.

With constant $\beta = 1-(1-\alpha)^{\frac{1}{d+1}}$, sharp exponent, only known proof.

All five theorems generalize to families of sets with d-Leray nerves.

Replace "convex hull of X" by "intersection of all famly members containing X".

[Alon - Kalai - Matoušek - Meshulam, 2002]

Theorem. Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ such that every subfamily intersects in at most r connected components, each of which is acyclic. $L(N(\mathcal{F})) \leq rd - 1$.

$$\left.\begin{array}{r}\text{Helly-type theorem}\\ \text{Fractional Helly theorem}\\ \text{First selection lemma}\\ \text{Weak } \epsilon\text{-net}\end{array}\right\}\ \begin{array}{l}\text{for acyclic families with bounded number of}\\ \text{connected components of intersection}\end{array}$$

[Kalai-Meshulam 2008][Colin de Verdière-Ginot-G 2014]

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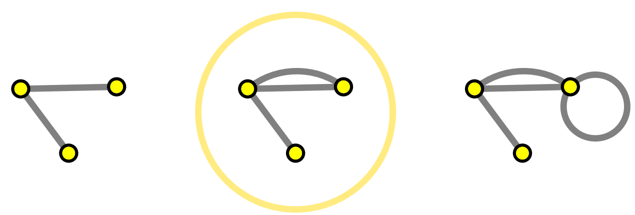
Fix the nerve by adding multiple simplices in case of multiple components

Simplicial poset: every lower-interval is isomorphic to a the face lattice of a simplex.



Can define (topological) geometric realization, simplicial homology...

Simplices no longer defined by their vertices; explicit incidences...

Arguments about co-faces may not extend

[Kalai-Meshulam 2008][Colin de Verdière-Ginot-G 2014]

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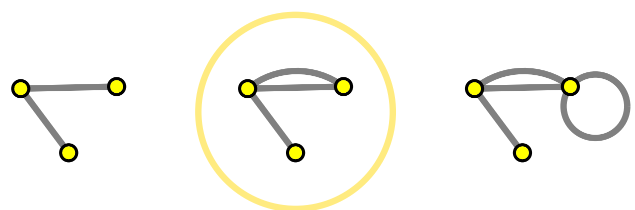

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Extend nerve theorem.

Analyze the removal of multiplicities $\simeq$ projection with fibers of bounded size.

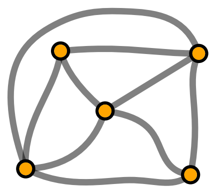
[Kalai-Meshulam 2008][Colin de Verdière-Ginot-G 2014]

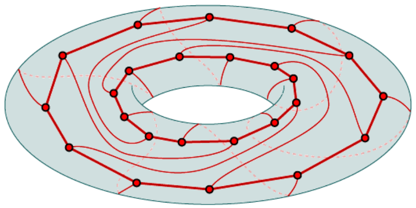
3. Some perspectives

Graphs are often used to model pairwise interactions, and hypergraphs are natural generalization to model many-wise interactions.

There is a rich theory of **geometric graphs**.

$$V - E + F = 2,\ E \leq 3V - 6,\ Cr(G) \geq \frac{E^3}{64V^2} \dots$$





The natural generalization of geometric graphs are **simplicial complexes**.

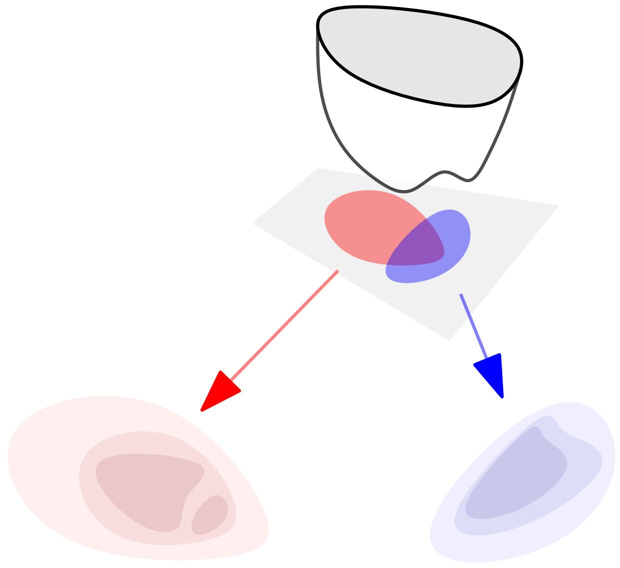
Non-planarity of K_5 $\rightsquigarrow$ Van Kampen-Flores theorem.

Conjecture. Let K be a d-dimensional simplicial complex on n vertices. If there exists an injection $|K| \to \mathbb{R}^{2d}$ then K has $O(n^d)$ simplices.

[Grünbaum][Kalai] [Sarkaria] [Dey]

Open for any $d \geq 2$.

Coming back to one of the original motivations...



Theorem. The Helly number of any family of subsets of $\mathbb{R}^d$ can be bounded by a function of d and

$$\max_{G \subseteq \mathcal{F}} \max_{i \leq \lceil \frac{d}{2} \rceil - 1} \widetilde{\beta}_i(\cap G, \mathbb{Z}_2).$$

$\Rightarrow$ Optimization problems with **bounded homological complexity** are "solvable" in linear time.

In terms of black boxes such as "minimize f over $O(1)$ of the constraints".

Level sets of constraints have bounded homology in dimension $< \lceil \frac{d}{2} \rceil$.

Thank you for your attention