Splitting Cycles in Triangulations

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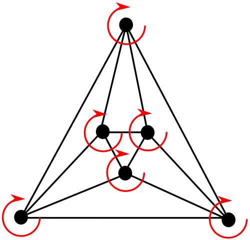
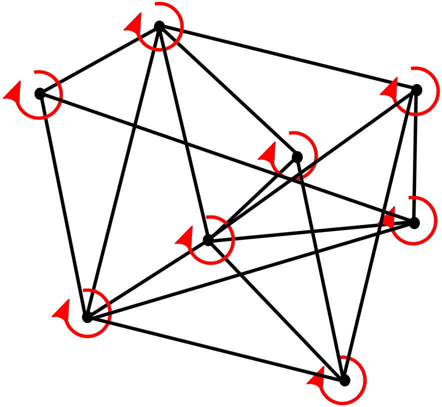
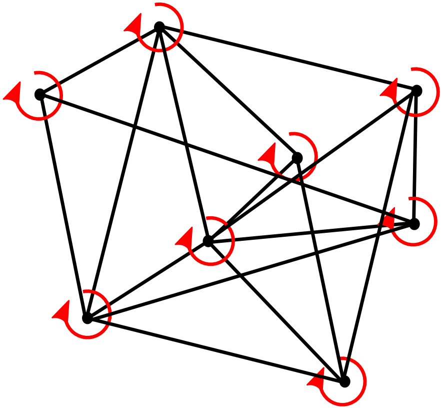
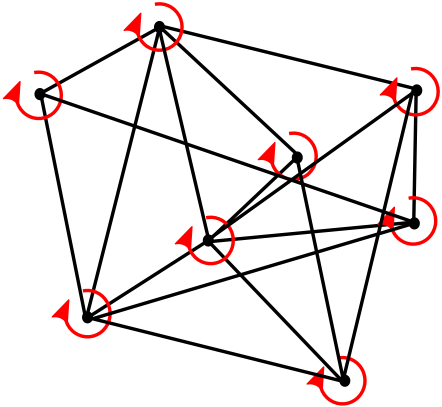






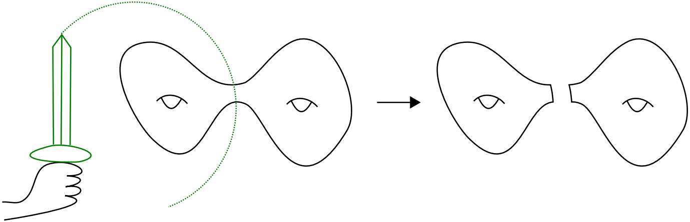
$$\chi(S) = V - E + F = 2 - 2g$$

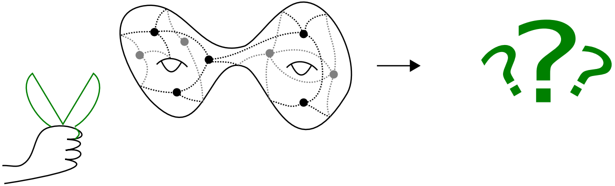


$$\chi(S) = V - E + F = 2 - 2g$$

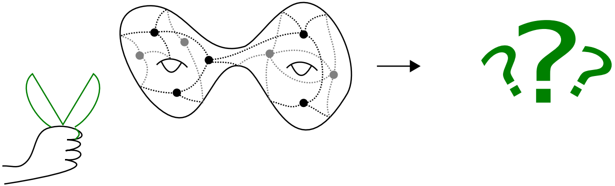
$$\chi(S) = 8 - 15 + 5 = -2 = 2 - 2 \cdot 2$$

Splitting Cycles





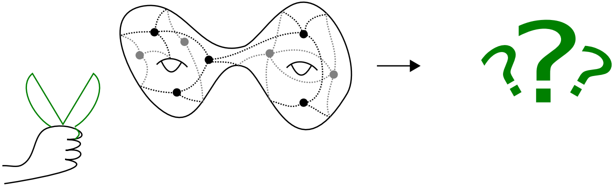
Splitting Cycles



Cabello et al. (2006)

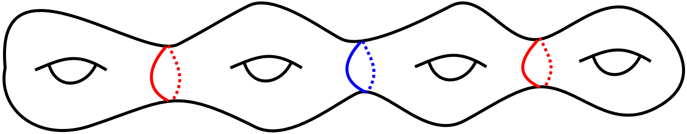
Deciding if a combiantorial map admits a splitting cycle is NP-complete.

Splitting Cycles



Barnette's Conjecture (1982)

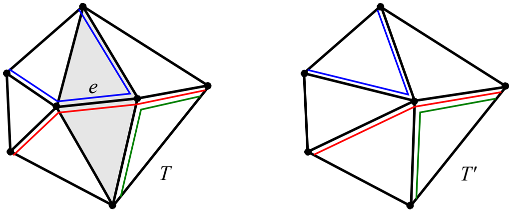
Every triangulations of surfaces of genus at least 2 admit a splitting cycle.



Conjecture (Mohar and Thomassen, 2001)

Every triangulations of surfaces of genus $g \geq 2$ admit a splitting cycle of every different type.

Irreducible Triangulations



- There are a finite number of irreducible triangulations of genus g. (Barnette and Edelson, 1988 and Joret and Wood, 2010)
- There are 396784 irreducible triangulations of genus 2.
- Unreachable for genus 3.

Genus 2 irreducible triangulations

First implementation by Thom Sulanke.

Genus 2:

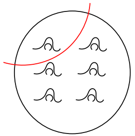
Number of triangulations: 396 784

n \ l	3	4	5	6	7	8	Average
10		2	51	681	130	1	6.09
11	2	58	2249	16138	7818	11	6.21
12	25	1516	20507	72001	22877	121	6.00
13	710	13004	50814	78059	16609	9	5.61
14	8130	30555	12308	3328	205	1	4.21
15	36794	1395	3	1	2		3.04
16	661	3					3.01
17	5						3.00

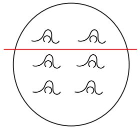
Genus 6

We consider the 59 non-isomorphic embeddings of K_{12}.
(Altshuler, Bokowski and Schuchert 1996)

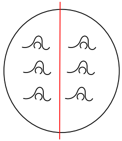
Average: 7.58
Worst-case: 8



Average: 9.41
Worst-case: 10



Average: 10.32
Worst-case: 12 (Hamiltonian cycle!)



Complete Graphs

Vertices: n

Edges: $n(n-1)/2$

Faces: $3f = 2e$ donc $f = 2/3 \cdot n(n-1)/2$

$$\chi(S) = v - e + f = n - \frac{n(n-1)}{2} + \frac{n(n-1)}{3} = 2 - 2g$$

$$g = \frac{(n-3)(n-4)}{12}$$

$$(n-3)(n-4) \equiv 0[12] \Leftrightarrow n \equiv 0, 3, 4 \text{ or } 7[12]$$

Ringel and Youngs, ∼1970

K_n can triangulate a surface if and only if $n \equiv 0, 3, 4$ or $7[12]$.

Type \ K_n	K_{15}	K_{16}	K_{19}	K_{27}	K_{28}	K_{31}	K_{39}	K_{40}	K_{43}
1	8	10	11	12	12	8	12	10	8
2	11	12	14	16	17	13	15	15	11
3	12	14	16	19	18	15	20	18	12
4	13	16	18	20	$\perp$	17	24	19	15
5	14	16	$\perp$	27	$\perp$	20	26	24	18
6		16	$\perp$	$\perp$	$\perp$	21	30	26	20
7			$\perp$	$\perp$	$\perp$	23	32	28	21
8			$\perp$	$\perp$	$\perp$	24	$\perp$	30	23
9			$\perp$	$\perp$	$\perp$	28	$\perp$	33	24
10			$\perp$	$\perp$	$\perp$	28	$\perp$	35	25
11				$\perp$	$\perp$	29	$\perp$	36	27
12				$\perp$	$\perp$	$\perp$	$\perp$	38	29
13				$\perp$	$\perp$	$\perp$	$\perp$	40	30
14				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	31
$\vdots$				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\vdots$
29						$\perp$	$\perp$	$\perp$	42
30						$\perp$	$\perp$	$\perp$	$\perp$
max type	5	6	10	23	25	31	52	55	65

$\perp$ = No cycle found.

Counter-Examples

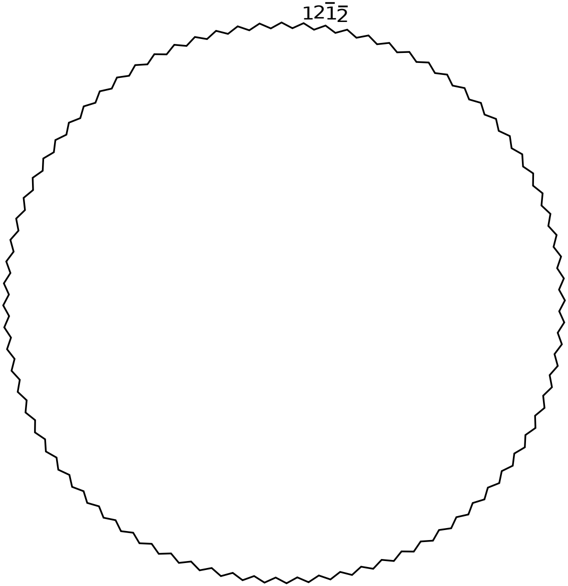
Mohar and Thomassen conjecture is false.

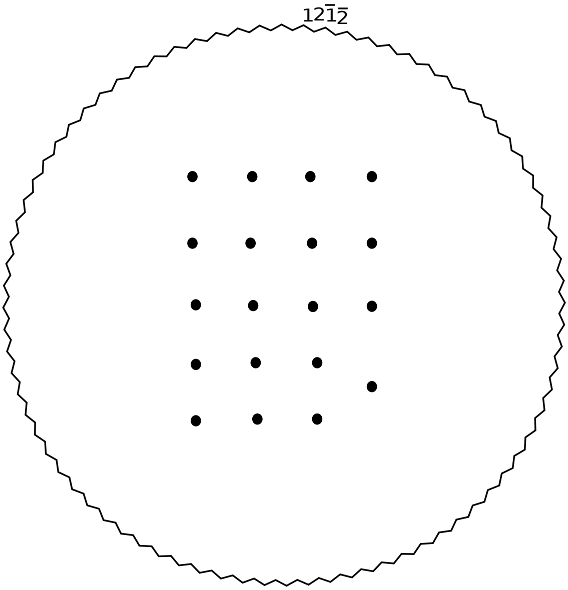
Type \ K_n	K_{15}	K_{16}	K_{19}	K_{27}	K_{28}	K_{31}	K_{39}	K_{40}	K_{43}
1	8	10	11	12	12	8	12	10	8
2	11	12	14	16	17	13	15	15	11
3	12	14	16	19	18	15	20	18	12
4	13	16	18	20	$\perp$	17	24	19	15
5	14	16	$\perp$	27	$\perp$	20	26	24	18
6		16	$\perp$	$\perp$	$\perp$	21	30	26	20
7			$\perp$	$\perp$	$\perp$	23	32	28	21
8			$\perp$	$\perp$	$\perp$	24	$\perp$	30	23
9			$\perp$	$\perp$	$\perp$	28	$\perp$	33	24
10			$\perp$	$\perp$	$\perp$	28	$\perp$	35	25
11				$\perp$	$\perp$	29	$\perp$	36	27
12				$\perp$	$\perp$	$\perp$	$\perp$	38	29
13				$\perp$	$\perp$	$\perp$	$\perp$	40	30
14				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	31
⋮				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	⋮
29						$\perp$	$\perp$	$\perp$	42
30						$\perp$	$\perp$	$\perp$	$\perp$
max type	5	6	10	23	25	31	52	55	65

$\perp$ = No cycle found.

Conjecture

For every $\alpha > 0$, there exists a triangulation with no splitting cycles of type larger than $\alpha \cdot \frac{g}{2}$.


$12\bar{1}\bar{2}$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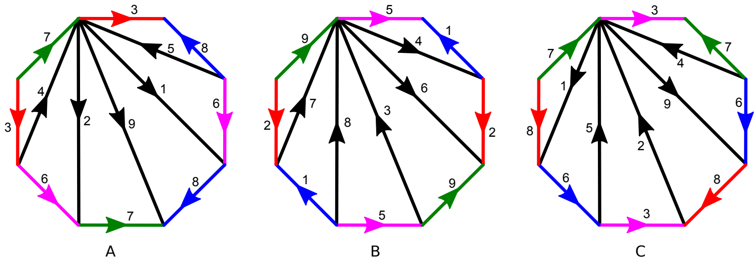

$$A//0 : 12, 15, 2, 9, 1, 14, 3, 7, 17, 13, 18, 8, 5, 6, 4, 16, 11, 10$$

$$A//1 : 13, 16, 3, 10, 2, 15, 4, 8, 18, 14, 0, 9, 6, 7, 5, 17, 12, 11$$

$$B//0 : 10, 12, 11, 16, 6, 4, 5, 8, 1, 15, 2, 9, 3, 14, 18, 7, 17, 13$$

$$C//0 : 12, 1, 14, 17, 9, 15, 3, 5, 13, 10, 8, 7, 4, 6, 18, 11, 2, 16$$

Ringel and Youngs, $\sim$1970

The three constructions A, B and C lead to correct triangulations of K_{19} on S_{20}.

Lawrenchenko et al., 1994

Those three constructions lead to three non-isomorphic embeddings.

Korzhik and Voss, 2001

Given two such constructions with labels in $\mathbb{Z}/19\mathbb{Z}$, the embeddings are isomorphic if and only if the function defined as the transformation of one system to the other is an automorphism of $\mathbb{Z}/19\mathbb{Z}$.

Small cases:

- There is only one way to draw K_3 on the plane.
- There is only one way to draw K_4 on the plane.
- There is only one way to draw K_7 on the torus.
- There are 59 different ways to draw K_{12} on S_6 (Altshuler et al., 1996).
- There are 182,200 different ways to draw K_{12} on N_{12} (Ellingham and Stephens, 2005).
- There are 243,088,286 different ways to draw K_{13} on N_{15} (Ellingham and Stephens, 2005).

Korzhik and Voss, 2001

There are at least $2^{(n-7)/6}$ nonisomorphic embeddings of K_n for $n \equiv 7[12]$.

Ellingham and Stephens, 2005

There are at most $(e^{-1/2}n)^{n^2/3}$ nonisomorphic embeddings of K_n.

Grannelle and Knor, 2010

There are at least $n^{n^2(1/864+o(1))}$ nonisomorphic embeddings of K_n for an infinite number of n such that $n \equiv 7[12]$.