Counting graphs with marked subgraphs

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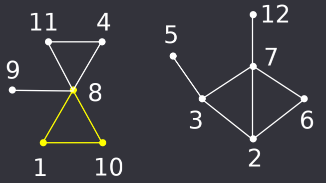
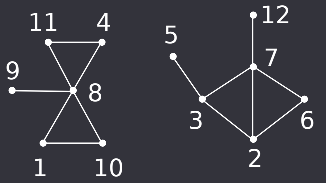
Problems

Counting graphs with one distinguished subgraph copy



Graph with 12 vertices, 14 edges, one distinguished triangle.

with a given number of subgraph copies



Graph with 12 vertices, 14 edges, 4 triangles.

Problems

Setting

F graph

$\mathcal{X}_F$ number of copies of F contained in a random graph, with n vertices and $m \sim cn^\alpha$ edges.

Problems

- find $\alpha^\star$ such that $\begin{cases} \mathcal{X}_F = 0 & \textit{a.a.s.} \text{ if } \alpha < \alpha^\star, \\ \mathcal{X}_F \geq 1 & \textit{a.a.s.} \text{ if } \alpha > \alpha^\star. \end{cases}$
- limit law of $\mathcal{X}_F$.

Resolution by Erdős and Rényi (1960), Bollobás (1981), Karoński and Ruciński (1983), Ruciński (1988); probabilistic approach.

Contribution: new approach based on analytic combinatorics (see the book of Flajolet and Sedgewick 2009).

Generating functions

The generating function (GF) associated to a graph family $\mathcal{F}$ is

$$F(z,w) = \sum_{G\in\mathcal{F}} w^{m(G)} \frac{z^{n(G)}}{n(G)!} = \sum_{n,m} \mathcal{F}_{n,m} w^m \frac{z^n}{n!}.$$

To investigate the number $\mathcal{F}_{n,m}$ of graphs from $\mathcal{F}$ with n vertices and m edges, we express its generating function $F(z,w)$

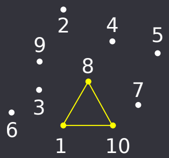
$$\mathcal{F}_{n,m} = n![z^n w^m]F(z,w).$$

	GF of the graph F	$w^{m(F)} \frac{z^{n(F)}}{n(F)!}$
Examples	GF of a set of isolated vertices	e^z
	GF of all graphs	$\sum_{n\geq 0}(1+w)^{\binom{n}{2}} \frac{z^n}{n!}$

Counting graphs where one copy of F is distinguished

Start with a copy of F and a set of isolated vertices.

$$F(z,w)\, e^z$$



Extract the coefficient

$$\mathrm{SG}^F_{n,m} = n![z^n w^m] F\left(z, \frac{w}{1+w}\right) e^z (1+w)^{\binom{n}{2}}.$$

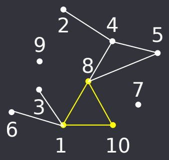
Also true when F is a family of graphs. In particular,

$$\mathbb{E}(\mathcal{X}_F) = \frac{\mathrm{SG}^F_{n,m}}{\mathrm{SG}_{n,m}} = \frac{n!}{\binom{\binom{n}{2}}{m}}[z^n w^m] F\left(z, \frac{w}{1+w}\right) e^z (1+w)^{\binom{n}{2}}.$$

Counting graphs where one copy of F is distinguished

Start with a copy of F and a set of isolated vertices.
Add some of the $\binom{n}{2} - m(F)$ available edges.

$$F(z, w)\, e^z (1 + w)^{\binom{n}{2} - m(F)}$$



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Counting graphs where one copy of F is distinguished

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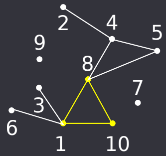
$$F\left(z, \frac{w}{1+w}\right) e^z (1+w)^{\binom{n}{2}}$$



Extract the coefficient

$$\mathrm{SG}_{n,m}^F = n![z^n w^m] F\left(z, \frac{w}{1+w}\right) e^z (1+w)^{\binom{n}{2}}.$$

Also true when F is a family of graphs. In particular,

$$\mathbb{E}(\mathcal{X}_F) = \frac{\mathrm{SG}_{n,m}^F}{\mathrm{SG}_{n,m}} = \frac{n!}{\binom{\binom{n}{2}}{m}} [z^n w^m] F\left(z, \frac{w}{1+w}\right) e^z (1+w)^{\binom{n}{2}}.$$

Asymptotics of $\mathbb{E}(\mathcal{X}_F)$

Regularity conditions on $F(z,w)$ and saddle-point method

$$\mathrm{SG}^F_{n,m} = n![z^n w^m]F\left(z, \frac{w}{1+w}\right) e^z (1+w)^{\binom{n}{2}} \approx F\left(n, \frac{m}{\binom{n}{2}}\right)\binom{\binom{n}{2}}{m}.$$

In particular, $\mathbb{E}(\mathcal{X}_F) \approx F\left(n, \frac{m}{\binom{n}{2}}\right)$.

Since $m \sim cn^\alpha$,

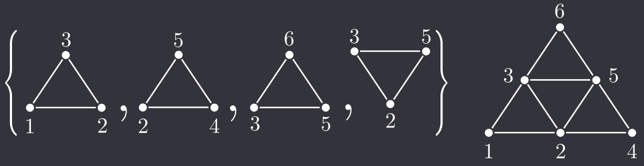
$$F\left(n, \frac{m}{\binom{n}{2}}\right) = \left(\frac{m}{\binom{n}{2}}\right)^{m(F)} \frac{n^{n(F)}}{n(F)!} = \Theta\left(n^{(\alpha-2)m(F)+n(F)}\right).$$

Thus, $\mathcal{X}_F = 0$ almost surely for $\alpha < 2 - \frac{n(F)}{m(F)}$.

Graphs with all copies of F marked

$\mathrm{SG}_{n,m}(u)$: GF of graphs where each subgraph F is marked by u.

Patchwork: set of copies of F that might share vertices and edges.
Generating function $P(z, w, u)$.



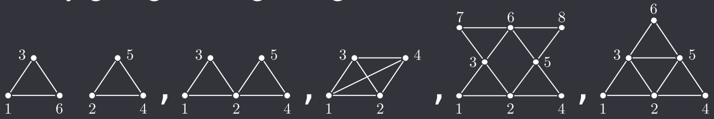
Inclusion-exclusion: consider $\mathrm{SG}_{n,m}(u+1)$.
Now each subgraph is either marked or left unmarked.
By definition, the marked subgraphs form a patchwork

$$\mathrm{SG}_{n,m}(u+1) = n![z^n w^m]P\left(z, \frac{w}{1+w}, u\right) e^z (1+w)^{\binom{n}{2}}.$$

Generating function of patchworks

Difficulty: we don't know the GF of patchworks!
(except for some very simple patterns)

Open problem: how many graphs with n vertices and m edges can be built by gluing t triangles together?



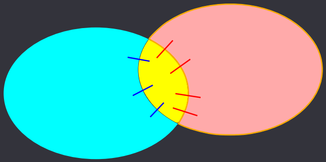
However, an approximation of the GF of patchworks is often enough for the asymptotics analysis.

Application: strictly balanced graphs

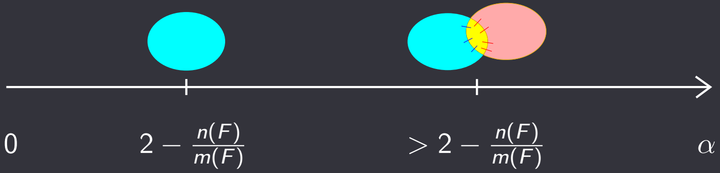
F is strictly balanced if all its strict subgraphs are less dense

$$\frac{m(F)}{n(F)} > \max_{H \subsetneq F} \frac{m(H)}{n(H)}.$$

In that case, any pair of non-disjoint copies has a higher density



so they typically do not appear for $m = \Theta\left(n^{2-\frac{n(F)}{m(F)}}\right)$



Application: strictly balanced graphs

Thus for $m \sim cn^{\alpha^\star}$, we need only consider disjoint patchworks

$$P(z, w, u) \approx e^{uF(z,w)}.$$

Graphs with n vertices, $m \sim cn^{\alpha^\star}$ edges, t subgraph copies of F

$$\begin{aligned} \mathrm{SG}_{n,m,t} &= n![z^n w^m u^t] P\left(z, \frac{w}{1+w}, u-1\right) e^z (1+w)^{\binom{n}{2}} \\ &\sim [u^t] e^{(u-1)F\left(n, m/\binom{n}{2}\right)} \binom{\binom{n}{2}}{m}. \end{aligned}$$

Thus, the limit law of $\mathcal{X}_F$ is Poisson $\left(\frac{(2c)^{m(F)}}{n(F)!}\right)$.

Application: graphs with degree constraints

Joint work with L. Ramos, work in progress with M. Mishna and F. Chyzak.

Problem: $D \subset \mathbb{Z}_{\geq 0}$, number $\mathrm{SG}_{n,m}^{D}$ of graphs with degrees in D.

Multigraphs: labelled oriented edges, loops and multiple edges.
Cut each edge into two labelled half edges ($\sim$ configuration model)

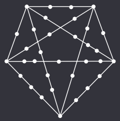
$$\mathrm{MG}_{n,m}^{D} = (2m)![x^{2m}]\left(\sum_{d \in D} \frac{x^d}{d!}\right)^n.$$

From multigraphs to graphs: remove the loops and double edges.
Each (n, m)-graph matches $2^m m!$ multigraphs.

Limit of the approach: the K_5 minors example

Fact: random graphs from $\mathrm{SG}_{n,n/2}$ contains a.s. no K_5 minor (see e.g. Janson, Knuth, Łuczak, Pittel 1993).

Can we prove it with our technique?



GF of K_5 minors $K(z,w) = \left(\frac{w}{1-zw}\right)^{\binom{5}{2}} \frac{z^5}{5!}$.
The expected number of K_5 minors is then

$$\frac{\mathrm{SG}_{n,n/2}^{K_5 \text{ minor}}}{\mathrm{SG}_{n,n/2}} = \frac{n!}{\binom{\binom{n}{2}}{n/2}} [z^n w^{n/2}] \left(\frac{w}{1-zw}\right)^{\binom{5}{2}} \frac{z^5}{5!} e^z (1+w)^{\binom{n}{2}} = \Theta\left(n^{\binom{5}{2}/2}\right).$$

Conclusion: in a random graph from $\mathrm{SG}_{n,n/2}$,

$$\mathbb{P}(\exists K_5\text{-minor subgraph}) \to 0, \quad \mathbb{E}(\# K_5\text{-minor subgraphs}) \to +\infty.$$

Conclusion

Results presented

- exact expression for the nb of graphs with a given nb of vertices, edges, and subgraphs copies of F,
- new proof of the limit law of $\mathcal{X}_F$ in the critical window when F is strictly balanced,
- marked subgraphs and degree constraints,

Other available results

- induced subgraphs,
- limit law of $\mathcal{X}_F$ outside the critical window.

In progress

- F is not strictly balanced,
- infinite family of forbidden subgraphs,
- phase transition of 2-SAT.

The GF of patchworks is known only for loops and double edges

$$\sum_{n \geq 0} \left(e^{-w} \sum_{k \geq 0} (1+u)^{\binom{k}{2}} \frac{w^k}{k!} \right)^{\binom{n}{2}} \frac{\left(z e^{uw/2}\right)^n}{n!} e^{-z}.$$