Periodic parallelogram polyominoes, and 321-avoiding affine permutations

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joint work with

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JCB 2017
January 26, 2017

321-avoiding permutations

A permutation $\sigma \in S_n$ is 321-avoiding if no integers $i < j < k$ are such that $\sigma(i) > \sigma(j) > \sigma(k)$

In S_6, $\sigma = 513624$ is not 321-avoiding while $\sigma = 231564$ is.

They are counted by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

The inversion number $inv(\sigma)$ is the number of inversions of the permutation σ i.e.

$$inv(\sigma) = |\{(i,j) \in [n]^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\}|.$$

For example $\sigma = 513624$ has 3+0+1+2+0=6 inversions.

Affine permutations

Definition (Affine permutations)

The group $\widetilde{S}_n$ is the set of permutations σ of $\mathbb{Z}$ satisfying

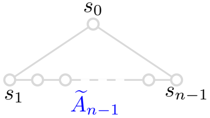
$$\sigma(i+n) = \sigma(i) + n \text{ and } \sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i.$$

Note that $\sigma(i) \equiv \sigma(j) \pmod{n}$ if and only if $i \equiv j \pmod{n}$.

An element of $\widetilde{S}_4$ is

$$\dots \mid 2, -7, -5, 4, \mid 6, -3, -1, 8, \mid 10, 1, 3, 12, \mid 14, 5, 7, 16 \mid \dots$$

- $\widetilde{S}_n$ is a Coxeter group of type $\widetilde{A}_{n-1}$;
- $s_0 = \cdots(-1-n, -n)(-1, 0)(-1+n, n)\cdots$;
- $s_i = \cdots(i, i+1)(i+n, i+1+n)\cdots$.



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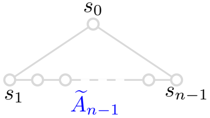

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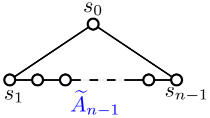

321-avoiding affine permutations

Definition

An affine permutation is 321-avoiding if there are not $i < j < k$ in $\mathbb{Z}$ such that $\sigma(i) > \sigma(j) > \sigma(k)$. We write $\sigma \in \widetilde{S}_n(321)$

For example

$$\ldots, \mid 2, -7, -5, 4, \mid \mathbf{6}, -\mathbf{3}, -\mathbf{1}, \mathbf{8}, \mid 10, 1, 3, 12, \mid 14, 5, 7, 16 \mid \ldots \in \widetilde{S}_4(321)$$

$$\ldots \mid -3, -5, 5 \mid \mathbf{0}, -\mathbf{2}, \mathbf{8} \mid 3, 1, 11 \mid 6, 4, 14 \mid 9, 7, 17 \mid \ldots \notin \widetilde{S}_3(321)$$

Definition (Affine inversions)

$$inv(\sigma) = |\{(i,j) \in [n] \times \mathbb{P} \mid i < j \text{ and } \sigma(i) > \sigma(j)\}|.$$

For example $inv([0, -2, 8]) = 7$.

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Generating function for permutations in $S_n(321)$

$$A_{n-1}(q) := \sum_{\sigma\in S_n(321)} q^{inv(\sigma)} \text{ and } A(x) = \sum_{n\geq 0} A_n(q)x^n.$$

Theorem (Barcucci, Del Lungo, Pergola, Pinzani, 2001)

We have $$A(x) = \frac{1}{1-xq} \times \frac{J(xq)}{J(x)},$$ *where*

$$J(x) := \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}$$

Here $(a)_0 := 1$

and $(a)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n\geq 1$

Generating function for affine permutations in $\tilde{S}_n(321)$

$$\widetilde{A}_{n-1}(q) := \sum_{\sigma \in \widetilde{S}_n(321)} q^{inv(\sigma)} \text{ and } \widetilde{A}(x) := \sum_{n \geq 1} \widetilde{A}_{n-1}(q) x^n$$

Theorem (B., Bousquet-Mélou, Jouhet, Nadeau, 2016)

We have

$$\widetilde{A}(x) = -x \frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}$$

Today's proof

Encode 321-avoiding (affine) permutations by :

1. (affine) alternating diagrams, then by
2. (periodic) parallelogram polyominoes, and finally by
3. (marked) heaps of segments.

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Motivations: fully commutative elements

The original goal was the computation of the series

$$\sum_{w \in W^{FC}} q^{\ell(w)}$$

where W^{FC} denotes the set of **fully commutative elements** in the Coxeter group W.

Theorem (Billey-Jockush-Stanley, 1993)

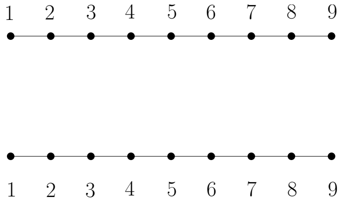
A permutation in S_n is fully commutative if and only if it is 321-avoiding.

Theorem (Green, 2001)

An affine permutation in $\widetilde{S}_n$ is fully commutative if and only if it is 321-avoiding.

From line diagrams to alternating diagrams

Take the 321-avoiding permutation $\sigma = 461279358 \in S_9$.



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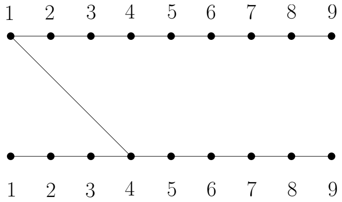

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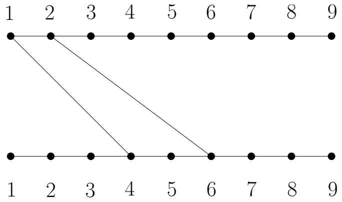

Alternating diagrams

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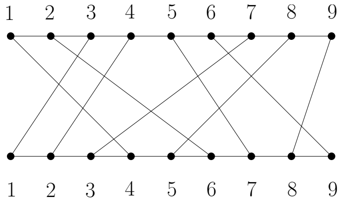

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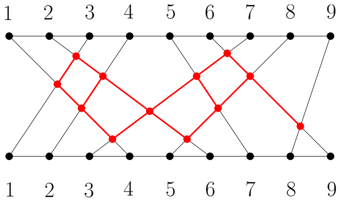
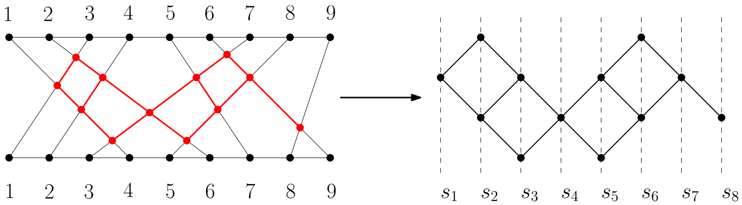


From line diagrams to alternating diagrams

The alternating diagram of $\sigma = 461279358 \in S_9$: $inv(\sigma) = 12$.



$S_n =< s_1, \ldots, s_{n-1} >$, where $s_i = (i, i+1)$.

Proposition (Characterization of alternating diagrams)

Alternating diagrams of S_n are characterized by:
(a) At most one occurrence of s_1 (resp. s_{n-1})
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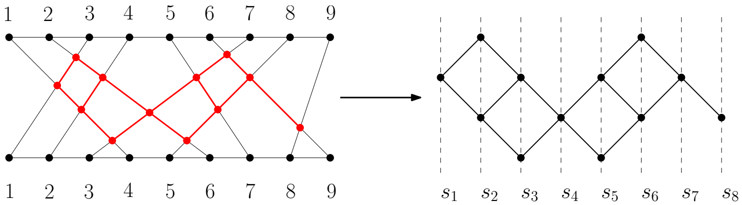

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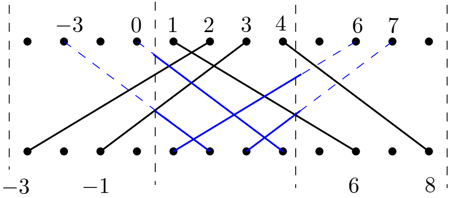
Here $\sigma \in \widetilde{S}_4(321)$ and $inv(\sigma) = 9$.



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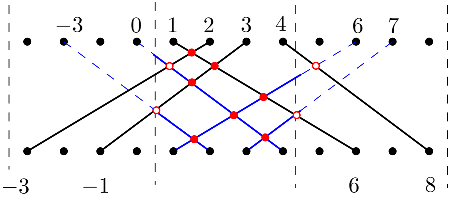

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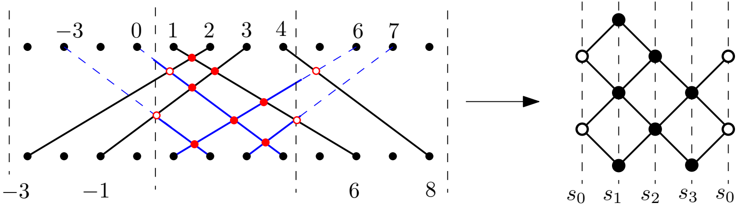

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- *Same number of occurrences of s_0 in the first and last column*
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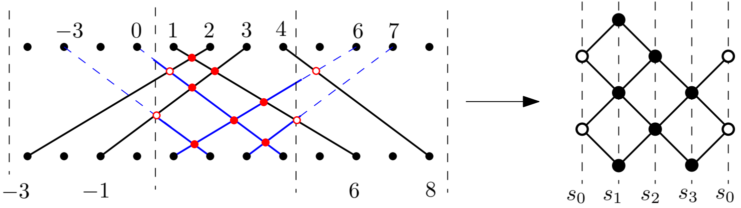

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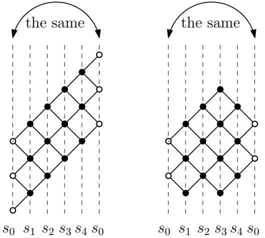
Affine alternating diagrams (on a cylinder)

Theorem (B, Jouhet, Nadeau, 2015)

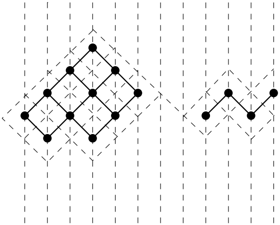
There is a bijection between $\widetilde{S}_n(321)$ and affine alternating diagrams of non-rectangular shape.



Rectangular shape

Affine alternating diagrams must be thought as "drawn on a cylinder"

From alternating diagrams to parellelogram polyominoes



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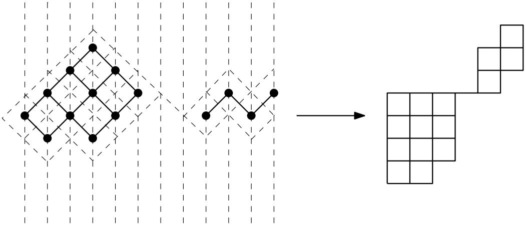

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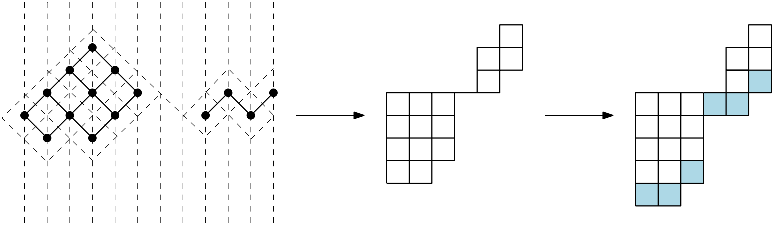
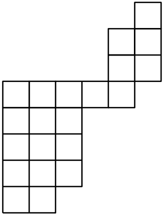


Theorem (Viennot, $<$ 1992)

There is a bijection between alternating diagrams and parallelogram polyominoes (PP).

A *PP* is a convex polyomino enclosed by two paths consisting of unit horizontal and vertical steps, both starting in the same point, ending in the same point, and non-intersecting elsewhere.

Bousquet-Mélou, Viennot encoding



$\longleftrightarrow$

(a_1, b_1) $\qquad$ (a_n, b_n)

$$(1, 5), (5, 5), (4, 4), (1, 1), (1, 3), (2, 3)$$

Convention: $a_1 = 1$

Parallelogram polyominoes are coded by sequences (a_i, b_i) with $a_1 = 1$, where:

- b_i is the height of the column C_i;
- a_i is the number of cells of C_i in contact with column C_{i-1}.

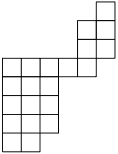
This set of finite sequences (a_i, b_i) satisfy

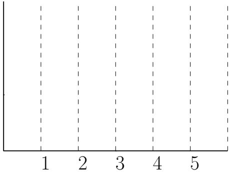
$$1 \le b_1 \ge a_2 \le b_2 \ge \ldots \le b_{n-1} \ge a_n \le b_n.$$

From parallelogram polyominoes to heaps of segments

Classical case (Bousquet-Mélou–Viennot, 1992)

$$(1,5),(5,5),(4,4),(1,1),(1,3),(2,3)$$





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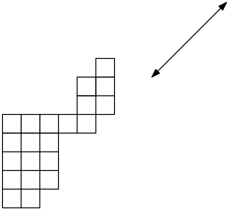

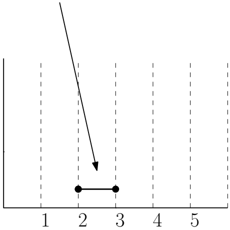

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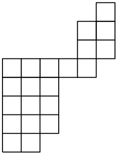

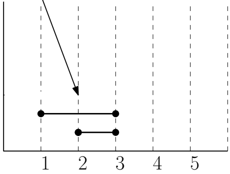

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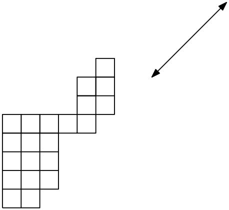

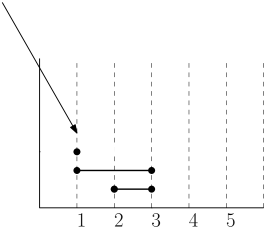

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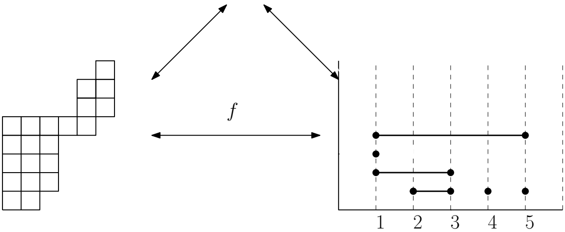


Half pyramid: a unique max $[1,5]$

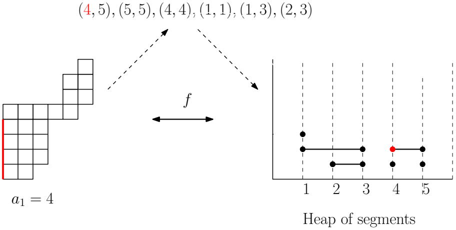
Theorem (Bousquet-Mélou–Viennot, 1992)

The map f is a bijection between the set of parallelogram polyominoes and the set of half pyramids of segments.

From $\mathcal{S}$ to heaps of segments

The set $\mathcal{S}$

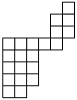
$$a_1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n.$$

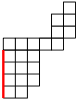


Proposition

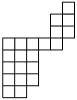
The map f extends to a bijection between the set of (PP) marked in their first column and the set of heaps of segments $\mathcal{H}$.

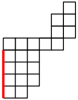
Toward periodic parallelogram polyominoes (PPP)

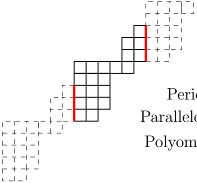
Parallelogram Polyominoes  $\longleftrightarrow$ $(\mathcal{S}, a_1 = 1)$ $\longleftrightarrow$ Half pyramids

Marked Parallelogram Polyominoes  $\longleftrightarrow$ $\mathcal{S}$ $\longleftrightarrow$ Heaps of segments

Toward periodic parallelogram polyominoes (*PPP*)

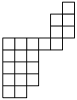
Parallelogram Polyominoes  $\longleftrightarrow$ $(\mathcal{S}, a_1 = 1)$ $\longleftrightarrow$ Half pyramids

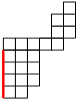
Marked Parallelogram Polyominoes  $\longleftrightarrow$ $\mathcal{S}$ $\longleftrightarrow$ Heaps of segments

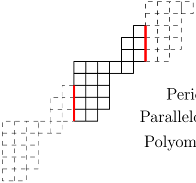
 Periodic Parallelogram Polyominoes $\longleftrightarrow$ $(\mathcal{S}, a_1 \leq b_n)$ $\longleftrightarrow$?

Recent papers (2016) on *PPP* also by Aval, Boussicault, Laborde–Zubieta, and Pétréolle.

Toward periodic parallelogram polyominoes (*PPP*)

Parallelogram Polyominoes  $\longleftrightarrow$ $(\mathcal{S}, a_1 = 1)$ $\longleftrightarrow$ Half pyramids

Marked Parallelogram Polyominoes  $\longleftrightarrow$ $\mathcal{S}$ $\longleftrightarrow$ Heaps of segments

 $\longleftrightarrow$ $(\mathcal{S}, a_1 \leq b_n)$ $\longleftrightarrow$?

Periodic Parallelogram Polyominoes

Recent papers (2016) on *PPP* also by Aval, Boussicault, Laborde–Zubieta, and Pétréolle.

Periodic parallelogram polyominoes (PPP)

Definition (PPP)

A **periodic parallelogram polyomino** is a a couple (P, a), where P is a marked PP and a is an integer between 1 and the height of the last column of P.

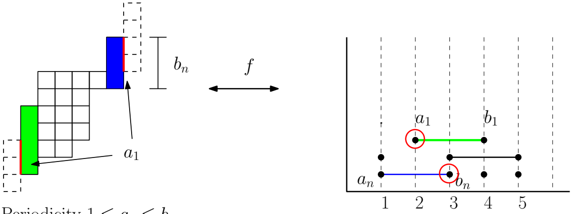
Periodic parellelogram polyominoes as sequences

These naturally correspond to sequences $(a_i, b_i)_{1 \leq i \leq n} \in \mathcal{S}$ such that $b_n \geq a_1$, i.e.

$$b_n \geq a_1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n.$$

From PPP to heaps of segments

$(2,4),(3,5),(5,5),(4,4),(1,1),(1,3)$



Periodicity $1 \leq a_1 \leq b_n$

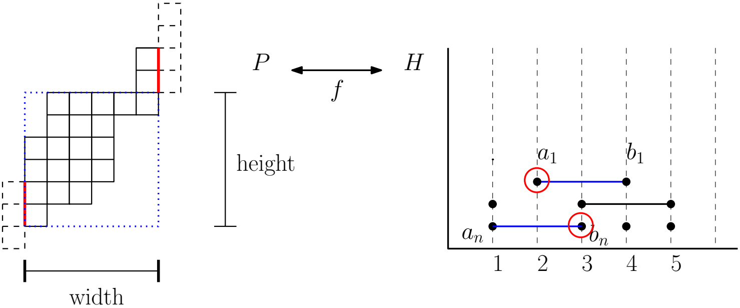
Condition $(\sim)$ on the heaps
read on left min et right max elements

Let $\widetilde{\mathcal{H}}$ be the set of heaps satisfying condition $(\sim)$.

Proposition

The map f induces a bijection between the set PPP and $\widetilde{\mathcal{H}}$.

Statistics on *PPP*



- width$(P) = |H|$ the number of segments in H
- height$(P) = \ell(H)$ the sum of the lengths of the segments of H
- area$(P) = e(H)$ sum of right endpoints of the segments of H

Generating functions for *PPP*

Goal: to compute the series

$$PPP(x,y,q) = \sum_{P \in PPP} x^{height(P)} y^{width(P)} q^{area(P)}.$$

or equivalently (after the bijection f)

$$\widetilde{\mathcal{H}}(x,y,q) = \sum_{H \in \widetilde{\mathcal{H}}} x^{\ell(H)} y^{|H|} q^{e(H)},$$

where

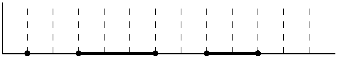
- $\ell(H)$ is the sum of the lengths of the segments of H;
- $|H|$ is the number of segments of H;
- $e(H)$ is the sum of right endpoints of the segments of H.

Generating functions for heaps of segments

$$\mathcal{H}(x,y,q) = \sum_{H \in \mathcal{H}} x^{\ell(H)} y^{|H|} q^{e(H)},$$

Trivial heaps $\mathcal{T}$

A trivial heap $T \in \mathcal{T}$ has no two pieces in concurrence.

$T =$ 

$$v(T) = x^5 y^3 q^{17}$$

Signed generating function for trivial heaps

$$\mathcal{T} = \mathcal{T}(x,y,q) = \sum_{T \in \mathcal{T}} (-1)^{|T|} x^{\ell(T)} y^{|T|} q^{e(T)}.$$

Generating functions for heaps of segments

Theorem (Inversion Lemma - Viennot, 1985)

$$\mathcal{H}(x,y,q) = \frac{1}{\mathcal{T}(x,y,q)} \text{ and } \mathcal{HP}(x,y,q) = \frac{\mathcal{T}^c(x,y,q)}{\mathcal{T}(x,y,q)}$$

where $\mathcal{T}$ (resp. $\mathcal{T}^c$) is the signed GF for trivial heaps (resp. not touching abscissa 1), and HP denotes the half pyramids.

Theorem (Bousquet-Mélou, Viennot, 1992)

$$\mathcal{T} = \sum_{n\geq 0} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_n (xq)_n} \text{ and } \mathcal{T}^c = \sum_{n\geq 1} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_{n-1} (xq)_n}.$$

Generating functions for heaps of segments

Theorem (Bousquet-Mélou, Viennot, 1992)

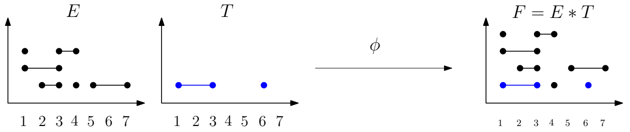
$$\mathcal{T} = \sum_{n\geq 0} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_n (xq)_n} \text{ and } \mathcal{T}^c = \sum_{n\geq 1} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_{n-1} (xq)_n} .$$

Since 321-avoiding permutations are in bijection with half pyramids, we obtain back the result of Barcucci et al, by setting $y \to y/q$ (recall that we added a box in each column), and then $y \to x$, in Viennot formula.

Theorem (Barcucci et al.)

$$A(x) = \frac{1}{1-xq} \times \frac{J(xq)}{J(x)} \text{ where } J(x) := \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}$$

Adaptation to our special heaps of segments in $\widetilde{\mathcal{H}}$



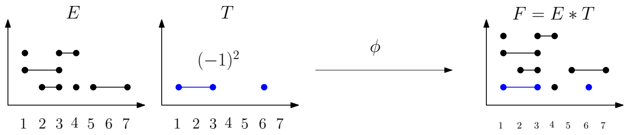
The idea is to study the pairs (F, T) belonging to $\phi(\widetilde{\mathcal{H}} \times \mathcal{T})$, where $\phi(E, T) = (E * T, T)$. We have that

Key Lemma

The pair (F, T) belongs to $\phi(\widetilde{\mathcal{H}} \times \mathcal{T})$ if and only if $T \subseteq \min(F)$ and one of the following cases occurs:

1. either $F \in \mathcal{T}$, and $|F \setminus T| = 1$.
2. or $F \notin \mathcal{T}$, $F \setminus U_1(F) \in \widetilde{\mathcal{H}}$ and $U_1(F) \subseteq T \subseteq U_2(F)$.

Adaptation to our special heaps of segments in $\widetilde{\mathcal{H}}$



$$\widetilde{\mathcal{H}} \times \mathcal{T} = \sum_{(F,U)\in\phi(\widetilde{\mathcal{H}}\times\mathcal{T})} v(F)(-1)^{|U|}. \quad (v(F) = x^{\ell(F)}y^{|F|}q^{e(F)})$$

By the previous lemma this is equal to

$$\sum_{F\in\mathcal{T}} |F|(-1)^{|F|-1}v(F) + \sum_{F,F\setminus U_1(F)\in\widetilde{\mathcal{H}}} v(F) \times \sum_{U_1(F)\subseteq U\subseteq U_2(F)} (-1)^{|U|}.$$

The first sum is $-y\partial_y\mathcal{T}$ while the second is always 0.

Adaptation to our special heaps of segments in $\widetilde{\mathcal{H}}$

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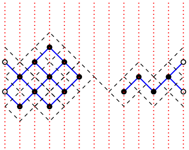
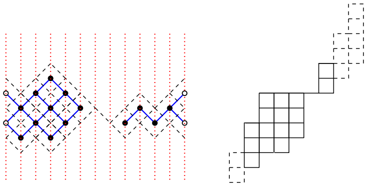
Theorem (B, Jouhet, Nadeau, 2016)

$$PPP(x,y,q) = -y\frac{\partial_y \mathcal{T}}{\mathcal{T}}.$$

Back to 321-avoiding affine permutations



From 321-avoiding affine permutations to PPP^*



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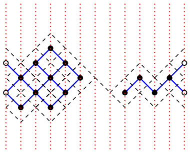

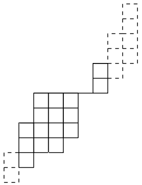

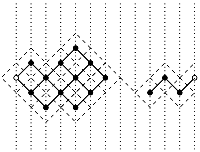

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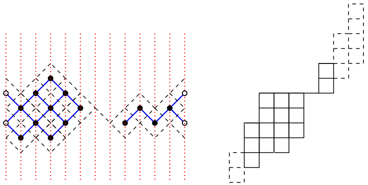

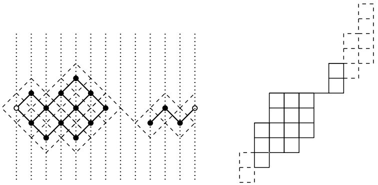

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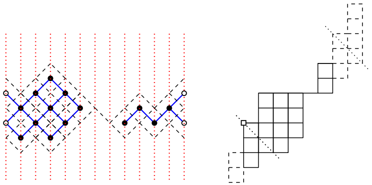
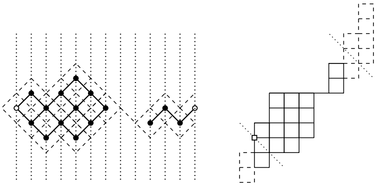
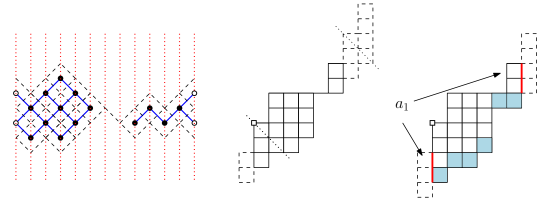


A mark in $[a_1, b_1]$
to recover s_0

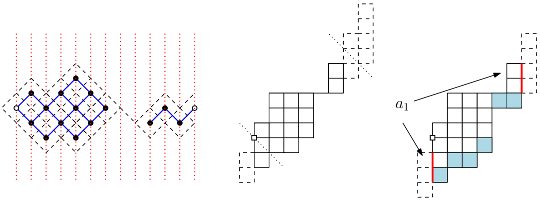


From 321-avoiding affine permutations to PPP^*

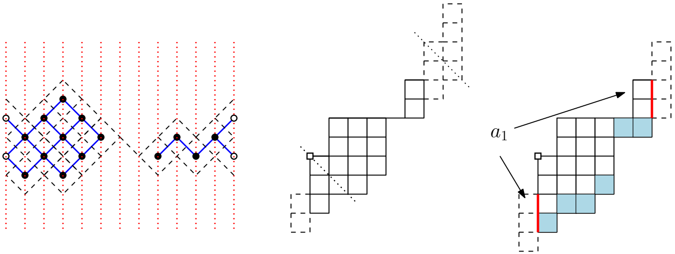


A mark in $[a_1, b_1]$ to recover s_0

The periodicity identifies a mark a_1 in $[1, b_n]$



From 321-avoiding affine permutations to PPP^*



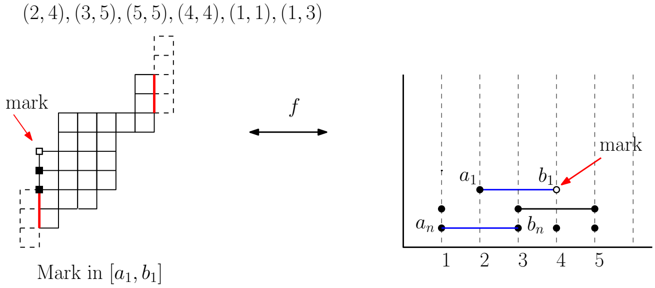
A mark in $[a_1, b_1]$ to recover s_0

The periodicity identifies a mark a_1 in $[1, b_n]$

Theorem (B, Jouhet, Nadeau, (2016))

There is a bijection between 321-avoiding affine permutations and *marked PPP* *of non-rectangular shape.*

Marked PPP



Definition

$PPP^* = PPP$ with a mark between a_1 and b_1 ;
$\widetilde{\mathcal{H}}^* =$ heaps in $\widetilde{\mathcal{H}}$ with a mark in their rightmost maximal segment.

Corollary

The map f induces a bijection between PPP^* and $\widetilde{\mathcal{H}}^*$.

Summary

Theorem (B, Jouhet, Nadeau, 2016)

$$PPP(x,y,q) = -y\frac{\partial_y \mathcal{T}}{\mathcal{T}} \qquad PPP^*(x,y,q) = -x\frac{\partial_x \mathcal{T}}{\mathcal{T}}.$$

Since marked PPP^* (minus those of rectangular shape) are in bijection with 321-avoiding affine permutations, we obtain (after taking care about the weight, $y \to y/q$, and $y \to x$) that

Theorem (B., Bousquet-Mélou, Jouhet, Nadeau)

$$\widetilde{A}(x) = -x\frac{J'(x)}{J(x)} - \sum_{n\geq 1}\frac{x^n q^n}{1-q^n}$$

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$$\widetilde{A}(x) = -x\frac{J'(x)}{J(x)} - \sum_{n\geq 1}\frac{x^n q^n}{1-q^n}$$

Open problem : pyramids

Denote by Π the set of pyramids (heaps with a **unique maximal element**).

By using the bijection ϕ we find

$$\sum_{H \in \Pi} x^{\ell(H)} y^{|H|} q^{e(H)} = -y \frac{\partial_y T}{T} = \sum_{H \in \widetilde{\mathcal{H}}} x^{\ell(H)} y^{|H|} q^{e(H)}$$

A bijection between the set $\widetilde{\mathcal{H}}$ and the set of pyramids Π would be nice, as would be a direct way of encoding periodic parallelogram polyominoes as pyramids.

Open problem : 321-avoiding involutions in S_n and $\widetilde{S}_n$

$$\mathcal{A} = \sum_{n\geq 0} \mathcal{A}_n^{Invo}(q)x^n \qquad \text{and} \qquad \tilde{\mathcal{A}} = \sum_{n\geq 1} \tilde{\mathcal{A}}_{n-1}^{Invo}(q)x^n.$$

Theorem (B., Bousquet-Mélou, Jouhet, Nadeau, 2016)

We have

$$\mathcal{A} = \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)} \qquad \textit{and} \qquad \tilde{\mathcal{A}} = -x\,\frac{\mathcal{J}'(x)}{\mathcal{J}(x)}, \qquad \textit{where}$$

$$\mathcal{J}(x) = \sum_{n\geq 0} \frac{(-1)^{\lceil n/2\rceil} x^n q^{\binom{n}{2}}}{((q^2))_{\lfloor n/2\rfloor}}.$$

Give a proof of this results using PPP^*.

End of the talk

The end