An application of Lovász local lemma in Symbolic Dynamics

Journées de Combinatoire de Bordeaux

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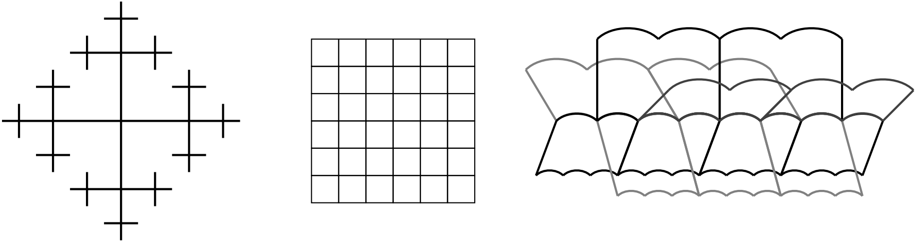
Outline of the talk.

1. Symbolic Dynamics and subshifts
2. Strongly aperiodic subshifts
3. Lovász Local Lemma in Symbolic Dynamics

Configurations, patterns and cylinders

- Let A be a finite alphabet, G be an infinite finitely generated group.
- Colorings $x : G \to A$ are called **configurations**.
- A **pattern** is a finite configuration $p : S \to A$.

Examples:



The Cantor space A^G?

- Endowed with the pro-discrete topology A^G is a **compact** and **metrizable** set.
- **Cylinders** form a clopen basis

$$[a]_g = \left\{x \in A^G \mid x_g = a\right\}.$$

- A **pattern** is a finite intersection of cylinders, or equivalently a finite configuration $p : S \to A$
- A **metric** for the cylinder topology is

$$d(x,y) = 2^{-\inf\{|g| \mid g \in G:\ x_{\mathbf{g}} \neq y_{\mathbf{g}}\}},$$

where $|g|$ is the length of the shortest path from 1_G to g in $\Gamma(G,S)$.

Subshifts: topological definition

The **shift** σ is the natural action of G on A^G by translation:

$$\sigma^g(x)_h = x_{g^{-1}\cdot h} \text{ for all } x \in A^G.$$

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$$\sigma^g(x)_h = x_{g^{-1}\cdot h} \text{ for all } x \in A^G.$$

(Topological) Definition: subshift

A **subshift** is a closed and σ-invariant subset of A^G.

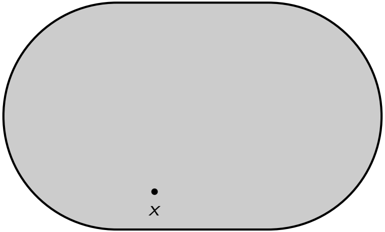
Examples:

- $X = \{x \in \{0,1\}^{\mathbb{Z}} \mid \text{no two consecutive 1's in } x\}$
- $X = \{x \in \{0,1\}^{\mathbb{Z}} \mid \text{finite blocks of 1's are of prime length}\}$
- $X = \{x \in \{0,1\}^{G} \mid \text{finite CC of 1's are of even length}\}$
- $X = \{x \in \{0,1\}^{G} \mid |\{g \in G : x_g = 1\}| \leq 1\}$

Discrete dynamical systems

(X, F) is a **discrete dynamical system** if:

- X is a topological compact space, called the **phase space**
- F is a continuous map : $X \to X$



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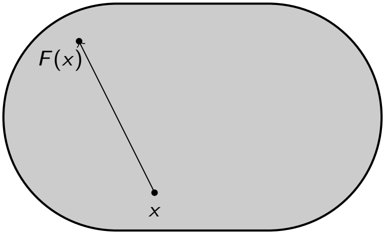

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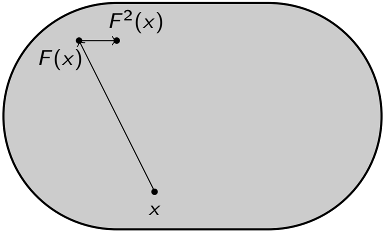

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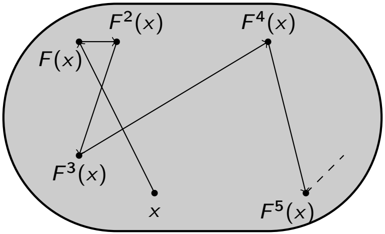
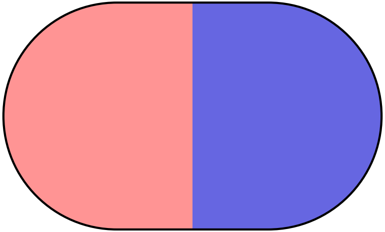


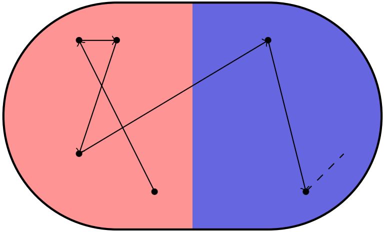
Coding of the orbits

- $X = \bigcup_{i=1}^{n} X_i$ a partition of the phase space X
- a color a_i associated with each X_i



Coding of the orbits

- $X = \bigcup_{i=1}^{n} X_i$ a partition of the phase space X
- a color a_i associated with each X_i
- orbit $(F^n(x))_{n\in\mathbb{N}}$ coded by a sequence $y \in \{a_1, \ldots, a_n\}^{\mathbb{N}}$



$\ldots$

Dynamical Systems and subshifts

- If the dynamical system (X, F) is *invertible* and *expansive*, then $X = \bigcup_{i=1}^{n} X_i$ can be chosen so that the set of coding of orbits

$$\widetilde{X} = \left\{ y \in \{a_1, \ldots, a_n\}^{\mathbb{Z}} \mid \exists x \in X, \forall k \in \mathbb{Z},\ F^k(x) \in X_i \right\}$$

  is a subshift in **one-to-one correspondence** with (X, F).

- Dynamical properties of the original system (X, F) can be read on the corresponding subshift $(\widetilde{X}, \sigma)$.

Remark: Works the same for G-actions.

Subshifts: combinatorial definition

(Combinatorial) Definition: subshift

Let F be a set of finite patterns. The **subshift** defined by the set of forbidden patterns F is the set

$$X_F = \left\{x \in A^G, \text{no pattern of } F \text{ appears in } x\right\}.$$

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Proposition

The topological and combinatorial definitions coincide.

Compactness of A^G

Proposition

The Cantor space A^G is compact for the pro-discrete topology.

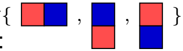
Concretely:

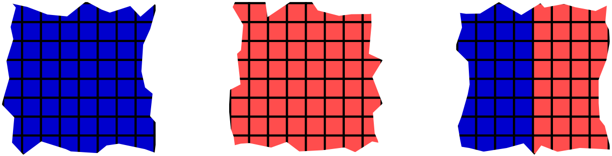
- X_F is non-empty iff there exist *arbitrarily big* finite patterns that avoid F (for instance every ball B_n can be colored avoiding F).
- If there exist *arbitrarily big* finite patterns with no letter a that avoid F, there exists a configuration in X_F with no letter a.
- You cannot force one letter a to appear exactly once in every configuration of a subshift.

Classes of subshifts: Subshifts of finite type (SFT)

Definition: subshift of finite type (SFT)

A **subshift of finite type (SFT)** is a subshift that can be defined by a finite set of forbidden patterns.

Example: The SFT X  contains the following configurations (in 2D):

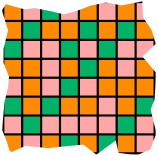


- simplest class with respect to the combinatorial definition
- 2D-SFT $\equiv$ Wang tilings 

Classes of subshifts: Sofic subshifts

A factor map $\Phi : A^G \to B^G$ is given by a local map ϕ (or equivalently Φ is a continuous and σ-commuting map):

$x \in A^{\mathbb{Z}^2}$ $\qquad\qquad$ $\Phi(x) \in B^{\mathbb{Z}^2}$



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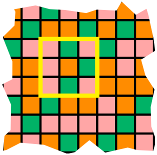

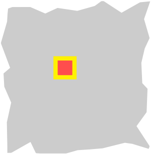

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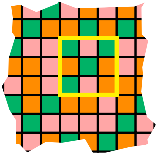

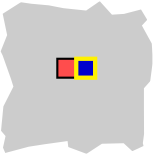

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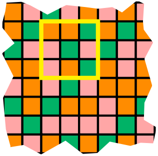

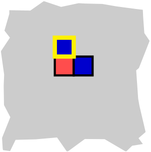

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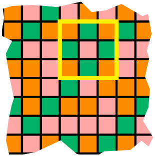

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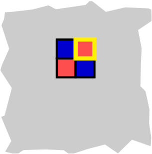

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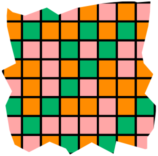

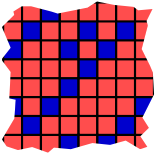

Definition: sofic subshift

A **sofic subshift** is the factor of an SFT.

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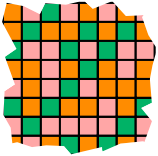

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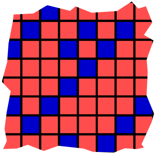

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A **sofic subshift** is the factor of an SFT.

- Recodings of SFT, with local neighborhood.
- In 1D ($G = \mathbb{Z}$), sofic subshifts are exactly those recognized by finite automata.

Examples

Which subshifts are SFTs? sofic subshifts?

Examples:

- $X = \left\{x \in \{0,1\}^{\mathbb{Z}} \mid \text{no two consecutive 1's in } x\right\}$
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  SFT
- $X = \left\{x \in \{0,1\}^{\mathbb{Z}} \mid \text{finite blocks of 1's are of prime length}\right\}$
  neither SFT nor sofic
- $X = \left\{x \in \{0,1\}^{G} \mid \text{finite CC of 1's are of even length}\right\}$
  Sofic but not SFT
- $X = \left\{x \in \{0,1\}^{G} \mid |\{g \in G : x_g = 1\}| \leq 1\right\}$
  Sofic on $\mathbb{Z}$, $\mathbb{Z}^2$, free groups, etc... but not SFT

In general, difficult to prove that a subshift is not sofic (except for $G = \mathbb{Z}$).

Outline of the talk.

1. Symbolic Dynamics and subshifts
2. Strongly aperiodic subshifts
3. Lovász Local Lemma in Symbolic Dynamics

Strongly aperiodic subshifts (I)

A subshift $X \subset A^G$ is **strongly aperiodic** if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

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Every 1D non-empty SFT contains a periodic configuration.

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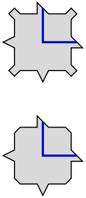
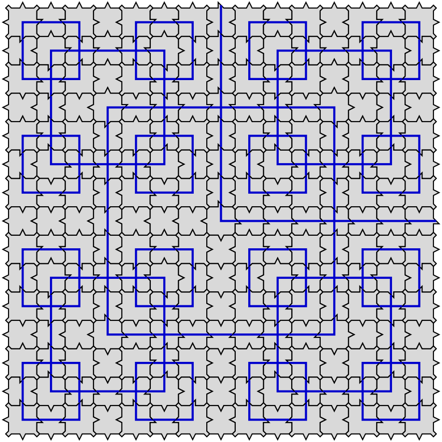
Every 1D non-empty SFT contains a periodic configuration.

Theorem (Berger 1966, Robinson 1971, Kari & Culik 1996, Jeandel & Rao 2015)

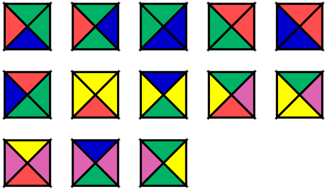
There exist strongly aperiodic SFTs on $\mathbb{Z}^2$.

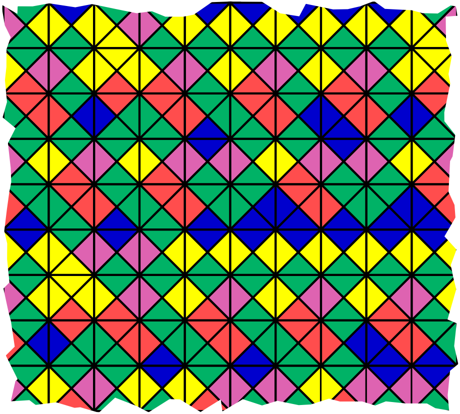
Robinson's SFT





Kari & Culik SFT





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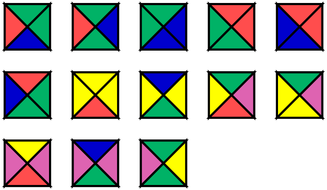

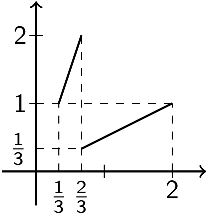

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- groups $\mathbb{Z}^2 \rtimes H$ where H has decidable **WP** (Barbieri & Sablik, 2016).

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Question (simpler)

Does every f.g. group admit strongly aperiodic subshifts?

Strongly aperiodic subshifts (III)

Theorem (Gao, Jackson & Seward, 2009)

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Theorem (A. Barbieri & Thomassé, 2015)

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Lovász Local Lemma

$$\left.\begin{array}{l}(A_i)_{i=1\dots n} \textbf{ mutually independent}\\ \text{Each } A_i \text{ can be avoided}\end{array}\right\} \Rightarrow A_1, \dots, A_n \text{ can be avoided.}$$

Proposition

If events $A_1, \dots, A_n$ are mutually independent, then

$$Pr\left(\bigcap_{i=1}^{n} \bar{A}_i\right) = \prod_{i=1}^{n} (1 - Pr(A_i)) .$$

What about the dependent case ?

Lovász Local Lemma

$$\left.\begin{array}{l}(A_i)_{i=1\ldots n} \textbf{\textit{ not very dependent}} \\ \text{Each } A_i \text{ can be avoided}\end{array}\right\} \Rightarrow A_1, \ldots, A_n \text{ can be avoided.}$$

Lovász Local Lemma (1975)

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$. For $A_i \in \mathcal{A}$, let $\Gamma(A_i)$ be the subset of $\mathcal{A}$ such that A_i is independent of the collection $\mathcal{A} \setminus (\{A_i\} \cup \Gamma(A_i))$. Suppose there are $x_i, \ldots, x_n$ such that $0 \leq x_i < 1$ and:

$$\forall A_i \in \mathcal{A} : Pr(A_i) \leq x_i \prod_{A_j \in \Gamma(A)} (1 - x_j)$$

then the probability of avoiding $A_1, A_2, \ldots, A_n$ is positive.

Lovász Local Lemma in Symbolic Dynamics (I)

How to use LLL in Symbolic Dynamics?

Suppose you want to prove that the subshift X is non-empty.

- Uniform Bernoulli measure on configurations space.
- Bad events $\approx$ forbidden patterns.
- Compactness + LLL (if applicable) show the non-emptiness of the subshift.

Lovász Local Lemma in Symbolic Dynamics (II)

Let G be a f.g. group, A a finite alphabet and μ the uniform Bernoulli probability measure on A^G.

A sufficient condition for being non-empty

Let $X \subset A^G$ be a subshift defined by $\mathcal{F} = \bigcup_{n \geq 1} \mathcal{F}_n$, where $\mathcal{F}_n \subset A^{B_{\boldsymbol{n}}}$. Suppose that there exists a function $x : \mathbb{N} \times G \to (0,1)$ such that:

$$\forall n \in \mathbb{N}, g \in G,\ \mu(A_{n,g}) \leq x(n,g) \prod_{\substack{gS_{\boldsymbol{n}} \cap hS_{\boldsymbol{k}} \neq \emptyset \\ (k,h) \neq (n,g)}} (1 - x(k,h)),$$

where $A_{n,g} = \left\{x \in A^G : x|_{gS_{\boldsymbol{n}}} \in \mathcal{F}_n\right\}$. Then the subshift X is non-empty.

Strong aperiodicity vs. the distinct neighborhood property

A subshift $X \subset A^G$ is **strongly aperiodic** if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

Fix $A = \{0, 1\}$.

A configuration $x \in \{0, 1\}^G$ has the **distinct neighborhood property** if for every $h \in G \setminus \{1_G\}$, there exists a finite $T \subset G$ s.t.

$$\forall g \in G,\ x_{|ghT} \neq x_{|gT}.$$

Strong aperiodicity vs. the distinct neighborhood property

A subshift $X \subset A^G$ is **strongly aperiodic** if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

Fix $A = \{0, 1\}$.

A configuration $x \in \{0,1\}^G$ has the **distinct neighborhood property** if for every $h \in G \setminus \{1_G\}$, there exists a finite $T \subset G$ s.t.

$$\forall g \in G,\ x_{|ghT} \neq x_{|gT}.$$

Proposition

If $x \in \{0,1\}^G$ has the distinct neighborhood property, then the subshift $\overline{Orb_\sigma(x)}$ is strongly aperiodic.

Distinct neighborhood property with LLL

Proposition

Every infinite f.g. group G has a configuration $x \in \{0,1\}^G$ with the distinct neighborhood property.

Proof:

- Take $(s_i)_{i\in\mathbb{N}}$ an enumeration of G with $s_0 = 1_G$.
- Choose $(T_i)_{i\in\mathbb{N}}$ a sequence of finite sets of G s.t.

$$T_i \cap s_i T_i = \emptyset \text{ and } |T_i| = Ci \text{ for some constant } C.$$

- $A_{n,g} = \left\{x \in \{0,1\}^G \mid x_{|gT_n} = x_{|gs_nT_n}\right\}$.
- Apply LLL with $x(n,g) = 2^{-\frac{Cn}{2}}$.

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Theorem

Every f.g. group G has a strongly aperiodic subshift on alphabet $\{0,1\}$.

An effectively closed strongly aperiodic subshift (I)

A subshift is G-**effectively closed** if it can be defined by a set of forbidden patterns recognizable by a Turing machine with oracle $\mathbf{WP}(G)$.

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Proposition

Let G a f.g. group and S a generating set. Then $\Gamma(G,S)$ has a square-free coloring with $2^{19}|S|^2$ colors.

An effectively closed strongly aperiodic subshift (II)

Theorem (A. Barbieri & Thomassé, 2015)

Every f.g. group G has a G-effectively closed strongly aperiodic subshift.

Sketch of the proof:

- Fix S and take $X \subset A^G$ be the subshift such that every square in $\Gamma(G,S)$ is forbidden.
- Let $g \in G$ such that $\sigma^g(x) = x$ for some $x \in X$.
- Factorize g as uwv with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.

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- If not, let $w = w_1 \dots w_n$ and consider the odd length walk $\pi = v_0 v_1 \dots v_{2n-1}$ on $\Gamma(G,S)$ defined by:

$$v_i = \begin{cases} 1_G & \text{if } i = 0 \\ w_1 \dots w_i & \text{if } i \in \{1, \dots, n\} \\ w w_1 \dots w_{i-n} & \text{if } i \in \{n+1, \dots, 2n-1\} \end{cases}$$

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- π is a path, and $x_{v_i} = x_{v_{i+n}} \Rightarrow g = 1_G$.

Conclusion

- Other applications of LLL? (challenge: find one *good* F)
- Which tools for subshifts on arbitrary f.g. group?
- Conjecture (Cohen + Jeandel): Every one-ended f.g. group with decidable **WP** has strongly aperiodic SFTs.

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Thank you for your attention !!

Example: the even shift

$X_{\texttt{even}} = \{x \in \{0,1\}^G |$ finite CC of 1's have even size $\}$.

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The even shift $X_{\texttt{even}}$ is sofic for every f.g. group G.

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Proof: Consider the G-SFT X_k, where $k = |B_1|$, with alphabet

$A_3 = \{$  $\}$ + rotations

$A_4 = \{$  $\}$ + rotations

$A_5 = \{$  $\}$ + rotations

$A_6 = \{$  $\}$ + rotations and reflections

etc...

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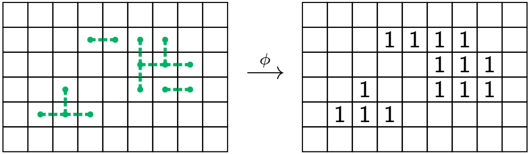
Green components have even size (handshaking lemma)$\Rightarrow \phi(X_k) \subseteq X_{\text{even}}$



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			1	1	1	1		
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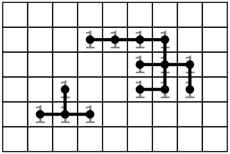

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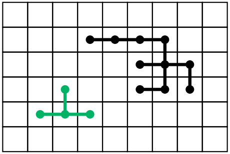

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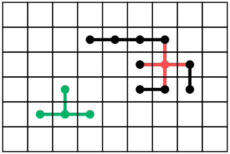

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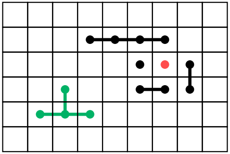

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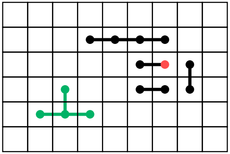

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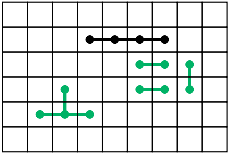

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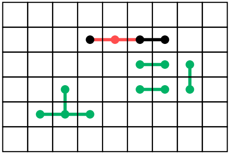

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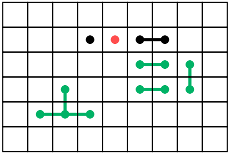

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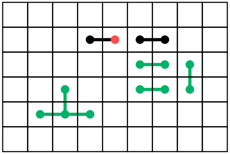

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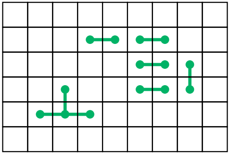

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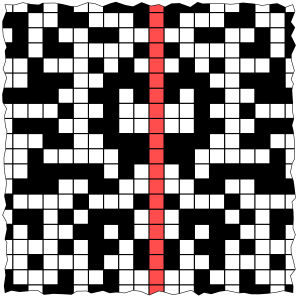
Mirror subshift in $\mathbb{Z}^2$

Let $A = \{ \square , \blacksquare , {\color{red}\blacksquare} \}$ and $X_{\text{mirror}} = X_{F_{\text{mirror}}} \subset A^{\mathbb{Z}^2}$ where

$$F_{\text{mirror}} = \left\{ \begin{array}{c}\square\\ {\color{red}\blacksquare}\end{array}, \begin{array}{c}\blacksquare\\ {\color{red}\blacksquare}\end{array}, \begin{array}{c}{\color{red}\blacksquare}\\ \square\end{array}, \begin{array}{c}{\color{red}\blacksquare}\\ \blacksquare\end{array} \right\} \cup \bigcup_{w \in A^*} \{ {\color{red}\blacksquare}\, w\, {\color{red}\blacksquare} , \blacksquare\, w\, {\color{red}\blacksquare}\, \tilde{w}\, \square , \square\, w\, {\color{red}\blacksquare}\, \tilde{w}\, \blacksquare \}$$

where $\tilde{w}$ denotes the mirror image of the word w.



The mirror subshift is not sofic